Plan More Examples (Countable-state case)

0. Questions
1. Extended Examples
2. Drift
3. Reversibility

Next Time: General-state Markov Chains

Midterm Exam: Tuesday 28 March in class
Homework 4 typo: #1 G&S *6.1*, 7–9.

Unless otherwise noted, let X be an irreducible, aperiodic Markov chain on countable state-space $\mathcal{S}$ with initial distribution μ and transition probabilities P.

Example 1. Renewal Process and Forward Recurrence Time Chain
Consider a process $Z = (Z_n)_{n\geq 0}$ given by

$$Z_n = Z_0 + \sum_{k=1}^{n} \Xi_k \tag{1}$$

for IID random variables Ξ_i and arbitrary Z_0. This is called a *renewal process.*

As an example, consider a critical part in a system that operates for some time and then fails. When it fails it is replaced. Think of Ξ as the lifetime of the replacement parts and Z_0 as the lifetime of the original part. Then, Z_n is the time of the nth replacement – a "renewal" of the system.

Given a renewal process, we can define $N_t = \sup\{n \geq 0\colon Z_n \leq t\}$. Then, N_t is a counting process that counts the renewals up to time t.

Suppose now that Z_0 and Ξ_1 take values in $\mathbb{Z}_+$ and have PMFs $\mu \equiv \mathsf{p}_{Z_0}$ and $\mathsf{p}_\Xi \equiv \mathsf{p}_{\Xi_1}$. Z_n is thus a countable-state Markov chain, but not a terribly interesting one as it marches inexorably toward ∞.

But we can define two related processes that will turn out to be very interesting in general. Define V^+ and V^- to be, respectively, the *forward* and *backward recurrence time* chains, as follows for $n \geq 0$:

$$V_n^+ = \inf\{Z_m - n\colon Z_m > n\} \tag{2}$$

$$V_n^- = \inf\{n - Z_m\colon Z_m \leq n\}. \tag{3}$$

Then V_n^+ represents the time until the next renewal, and V_n^- represents the time since the last renewal. These are sometimes also called the *residual lifetime* and *age* processes.

Are these Markov chains? What are the state spaces?

We can check the Markov Property explicitly. But the regeneration of the system at each renewal gives us a simple way of seeing it. When $V_n^+ > 1$, for instance, the next time is determined. When $V_n^+ = 1$, a renewal ensues and the next time is an independent waiting time.

The state space of V^+ is $\mathbb{Z}_+$; the state space of V^- is $\mathbb{Z}_\oplus$.

What are the transition probabilities?

If $V_n^+ = k > 1$, then $V_{n+1}^+ = k - 1$ by construction. If $V_n^+ = 1$, then a renewal occurs at time $n+1$, so the time until the following renewal has distribution ξ. Hence,

$$P(k, k-1) = 1 \quad \text{for } k > 1, \tag{4}$$

$$P(1, k) = \xi(k) \quad \text{for } k \in \mathbb{Z}_+. \tag{5}$$

For V^-, we can reason similarly. Let S_Ξ be the survival function of Ξ_1. Then,

$$P(k, k+1) = \mathsf{P}\{\Xi > k+1 \mid \Xi > k\} = \frac{\mathsf{S}_\Xi(k+1)}{\mathsf{S}_\Xi(k)} \tag{6}$$

$$P(k, 0) = \mathsf{P}\{\Xi = k+1 \mid \Xi > k\} = \frac{\mathsf{p}_\Xi(k+1)}{\mathsf{S}_\Xi(k)}. \tag{7}$$

Is V^+ irreducible? Is it recurrent?

If there exists an $M \in \mathbb{Z}_+$ such that $\mathsf{S}_\Xi(M) = 0$ and $\mathsf{p}_\Xi(M) > 0$, then all states $j > M$ are transient, and all states $\{1, \ldots, M\}$ communicate and are recurrent since there is only a finite number. (To see the latter, note that we can find a positive probability path between each pair of states in this set.)

If no such M exists, then V^+ is irreducible. Note that for all states $n > 1$, $P_{!1}^{n-1}(n, 1) = 1$. Hence,

$$R(1,1) = \sum_{n \geq 1} \mathsf{p}_\Xi(n) P_{!1}^{n-1}(n, 1) = 1. \tag{8}$$

So the chain is recurrent in this case as well.

What is the long-run behavior of the chain?

Let $\rho(j) = \sum_{n \geq 1} P_{!1}^n(1, j)$. Because $P_{!1}^n(1, j) = \mathsf{p}_\Xi(j + n - 1)$ for $n \geq 1$, we can write

$$\rho(j) = \sum_{n \geq 1} \mathsf{p}_\Xi(j + n - 1) = \sum_{n \geq j} \mathsf{p}_\Xi(n) = \mathsf{S}_\Xi(j - 1). \tag{9}$$

Notice that

$$\sum_{j \geq 1} \rho(j) P(j, k) = \rho(1)\mathsf{p}_\Xi(k) + \rho(k+1) \tag{10}$$

$$= \mathsf{p}_\Xi(k) + \mathsf{S}_\Xi(k) \tag{11}$$

$$= \rho(k). \tag{12}$$

This invariant measure is positive (on its support) and is finite if and only if

$$\sum_{n \geq 1} \rho(n) = \sum_{n \geq 1} \mathsf{S}_\Xi(n-1) = \sum_{n \geq 1} n\mathsf{p}_\Xi(n) = \mathsf{E}\Xi_1 < \infty. \tag{13}$$

In this case, $\pi(k) = \rho(k)/\mathsf{E}\Xi_1$ is a stationary distribution.

Idea 2. The above argument leads to an interesting idea for the countable case.

Suppose that X is recurrent, Pick a state $s_0 \in \mathcal{S}$ such that it is easy to compute $P^n_{!s_0}(s_0, s)$ for any $s \in \mathcal{S}$. Define

$$\rho(s) = \sum_{n \geq 1} P^n_{!s_0}(s_0, s). \tag{14}$$

Note that $\rho(s_0) = 1$ because the chain is recurrent (i.e., $R(s_0, s_0) = 1$). Then, mimicking the above argument, we get

$$\begin{aligned}
\sum_{s \in \mathcal{S}} \rho(s) P(s, s') &= \rho(s_0) P(s_0, s') + \sum_{s \neq s_0} \sum_{n \geq 1} P^n_{!s_0}(s_0, s) P(s, s') && (15)\\
&= P(s_0, s') + \sum_{n \geq 2} P^n_{!s_0}(s_0, s') && (16)\\
&= P_{!s_0}(s_0, s') + \sum_{n \geq 2} P^n_{!s_0}(s_0, s') && (17)\\
&= \rho(s'). && (18)
\end{aligned}$$

This is finite only if

$$\begin{aligned}
\rho(\mathcal{S}) &= \sum_{s \in \mathcal{S}} \sum_{n \geq 1} P^n_{!s_0}(s_0, s) && (19)\\
&= \sum_{n \geq 1} \sum_{s \in \mathcal{S}} P^n_{!s_0}(s_0, s) && (20)\\
&= \sum_{n \geq 1} P^n_{!s_0}(s_0, \mathcal{S}) && (21)\\
&= \sum_{n \geq 1} \mathsf{P}_{s_0}\{T_{s_0} \geq n\} && (22)\\
&= \mathsf{E}_{s_0} T_{s_0} && (23)\\
&< \infty, && (24)
\end{aligned}$$

which is just the positive recurrence condition.

Hence,

$$\pi(s) = \frac{\sum_{n \geq 1} P^n_{!s_0}(s_0, s)}{\sum_{n \geq 1} P^n_{!s_0}(s_0, \mathcal{S})} = \frac{1}{\mathsf{E}_{s_0} T_{s_0}}. \tag{25}$$

We already knew the last equality, but this gives a new way of finding a stationary distribution, using equation (14).

Example 3. And now, for something completely different
Define a Markov chain on $\{1, \ldots, 6\}$ with transition probability matrix

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 2/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{bmatrix} \tag{26}$$

Let's go to town.

What are the communicating classes? Are they each absorbing or not? Recurrent or transient? What is the limiting behavior of the chain corresponding to each class? What does this say about the long-run behavior of the chain itself? What can we say about the transient class? What is $\lim_{n\to\infty} P^n$*?*

Let D denote the set of states in non-absorbing communicating classes. Let $C_1, \ldots, C_m$ be the absorbing communicating classes. Let $h_r(i) = H(i, C_r)$ for $i \in \mathcal{S}$. By conditioning on the first step, we get for $i \in D$ taht

$$h_r(i) = \sum_{j \in C_r} P(i,j) + \sum_{j \in D} P(i,j) h_r(j). \tag{27}$$

h_r is a non-negative solution to this equation. It can be shown (by induction on time n) that h_r is the *smallest* non-negative solution of this equation.

Intuitively, what should $\lim_{n\to\infty} P^n(i,j)$ *be for* $i \in D$ *and* $j \in C_r$*, assuming* C_r *is aperiodic?*

Example 4. Upper Hessenberg Transition Probabilities

Suppose $\mathcal{S} = \mathbb{Z}_{\oplus}$ and suppose the non-zero transition probabilities are of the form

$$P(0,k) = a_k \qquad \text{for } k \geq 0, \tag{28}$$

$$P(j,k) = a_{k-j+1} \qquad \text{for } j \geq 1, k \geq j-1, \tag{29}$$

where $(a_n)_{n\geq 0}$ is a non-negative sequence satisfying $\sum_n a_n = 1$ and $\sum_n na_n < \infty$.

We want to determine the stability of this chain. Consider the function $\Delta(s)$ on the state space, given by

$$\Delta(s) = \mathsf{E}(X_{n+1} - X_n \mid X_n = s). \tag{30}$$

This tells us how much the chain "drifts" on average in one step when starting at s. Note that because we are dealing with a time-homogeneous chain,

$$\Delta(s) = \mathsf{E}_s X_1 - s = \sum_{s'} P(s,s')s' - s. \tag{31}$$

Let's compute this function.

$$\Delta(0) = \sum_k P(0,k)k \tag{32}$$

$$= \sum_k ka_k < \infty \tag{33}$$

$$\Delta(j) = \sum_k P(j,k)k - j \tag{34}$$

$$= \sum_{k=j-1}^{\infty} a_{k-j+1}k - j \tag{35}$$

$$= \sum_{k=j-1}^{\infty} a_{k-j+1}(k-j+1) + \sum_{k=j-1}^{\infty} a_{k-j+1}(j-1) - j \tag{36}$$

$$= \sum_{n=0}^{\infty} na_n - 1 < \infty, \tag{37}$$

for $j \geq 1$.

Case (i): $\sum_n na_n > 1$.

In this case, $\Delta(j) > 0$ for every j, hence on average in any state, we tend toward higher states. This "positive drift" seems to suggest (though does not prove) transience.

Case (ii): $\sum_n na_n < 1$.

In this case, $\Delta(0) > 0$ and $\Delta(j) < -\epsilon \equiv \sum_n na_n - 1$ for all $j \geq 1$. Hence, on average, whenever the chain is away from zero, it tends to move back toward zero. This suggests recurrence.

What we can we make of this?

Definition 5. Drift

Let V be a non-negative function on the state space (that is, $V : \mathcal{S} \to \mathbb{R}_{\oplus}$). Define the drift operator Δ_X by

$$\Delta_X V = PV - V. \tag{38}$$

That is,

$$(\Delta_X V)(s) = \sum_{s' \in \mathcal{S}} P(s,s')V(s') - V(s) = \mathsf{E}(V(X_{n+1}) - V(X_n) \mid X_n = s) = \mathsf{E}_s(V(X_1)) - V(s). \tag{39}$$

Note that, in general, $\Delta_X V$ takes values in $[-\infty, \infty]$. When the chain is understood, I'll use Δ instead of Δ_X.

Theorem 6. Foster's Drift Criterion

Suppose there exists a non-negative function $V : \mathcal{S} \to \mathbb{R}_{\oplus}$, an $\epsilon > 0$, and a finite set $S_0 \subset \mathcal{S}$ such that

$$|\Delta_X V(s)| < \infty \quad \text{for } s \in S_0 \tag{40}$$

$$\Delta_X V(s) \le -\epsilon \quad \text{for } s \notin S_0. \tag{41}$$

Then, X is positive recurrent.

Proof

For $s \in S_0$, $|\Delta_X V(s)| < \infty$ implies that $|PV| < \infty$ on S_0. Define

$$u^{[n]}(s) = \sum_{s'} P^n(s,s')V(s'), \tag{42}$$

for $n \ge 0$.

Notice that for $m \ge 0$,

$$\begin{aligned}
u^{[m+1]}(s) &= \sum_{s'} P^{m+1}(s,s')V(s') && (43)\\
&= \sum_{s'} \sum_{t \in \mathcal{S}} P^m(s,t)P(t,s')V(s') && (44)\\
&= \sum_{t \in \mathcal{S}} P^m(s,t) \sum_{s'} P(t,s')V(s') && (45)\\
&= \sum_{t \in \mathcal{S}} P^m(s,t)(PV)(t) && (46)\\
&= \sum_{t \in S_0} P^m(s,t)(PV)(t) \;+\; \sum_{t \notin S_0} P^m(s,t)(PV)(t) && (47)\\
&\le \sum_{t \in S_0} P^m(s,t)(PV)(t) \;+\; \sum_{t \notin S_0} P^m(s,t)(V(t) - \epsilon) && (48)\\
&\le \sum_{t \in S_0} P^m(s,t)((PV)(t) + \epsilon) \;+\; \sum_{t \in \mathcal{S}} P^m(s,t)(V(t) - \epsilon) && (49)\\
&= \sum_{t \in S_0} P^m(s,t)((PV)(t) + \epsilon) \;+\; u^{[m]}(s) - \epsilon. && (50)
\end{aligned}$$

This gives us an upper bound for $u^{[m+1]} - u^{[m]}$. Summing these together by telescoping gives

$$0 \le u^{[n+1]}(s) \le u^{[0]}(s) + \sum_{t \in \S_0} \sum_{m=0}^{n} P^m(s,t)(PV(t) + \epsilon) - (n+1)\epsilon. \tag{51}$$

Rearranging and dividing by $n+1$ gives

$$\frac{u^{[0]}(s)}{n+1} + \sum_{t \in \S_0} \left(\frac{1}{n+1} \sum_{m=0}^{n} P^m(s,t) \right) (PV(t) + \epsilon) \ge \epsilon. \tag{52}$$

Taking limits on both sides as $n \to \infty$ and using the fact that S_0 is finite and $0 \le PV(t) < \infty$ on S_0, gives us

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{m=0}^{n} P^m(s,t) > 0, \tag{53}$$

for sum $t \in S_0$. But it follows that (Césaro summation), that $\sum_n P^n(s,t)$ cannot converge. Hence, X is recurrent. Positive recurrence follows as well, though somewhat more delicately.

Example 4 cont'd Upper Hessenberg Transition Probabilities

We know from the above that $\sum_n na_n < 1$ implies that this chain is positive recurrence. What about when $\sum_n na_n \ge 1$? Can we deduce transience from this?

It turns out that Foster's criterion cannot simply be reversed. We can get the following.

Theorem 7. X is transient if and only if there exists a bounded, non-negative function V on $\mathcal{S}$ and $r \ge 0$ such that $\{s \in \mathcal{S} \colon V(s) > r\}$ and $\{s \in \mathcal{S} \colon V(s) \le r\}$ are both nonempty *and*

$$\Delta V(s) > 0 \qquad \text{if } V(s) > r. \tag{54}$$

(Recall that we have assumed here that X is an irreducible chain with countable state space.)

The proof of this relies on a rather cute result:

Lemma Let $C \subset S$. Let $h_*(s) = H(s, C)$ be the hitting probability of C from s. (Recall, the hitting time S_C is zero if the chain starts in C and is otherwise equal to T_C.)

Then, if $h \colon \mathcal{S} \to \mathbb{R}_\oplus$ is a solution to

$$\Delta h(s) \le 0 \text{ if } s \in C^c \tag{55}$$

$$h(s) \ge 1 \text{ if } s \in C \tag{56}$$

Then, $h* \le h$.

Now, to the theorem, suppose that $|V| \le M$. We must have, by the conditions, that $M > r$. (Why?) Define

$$h_V(s) = \begin{cases} 1 & \text{if } V(s) \le r \\ \frac{M - V(s)}{M - r} & \text{if } V(s) > r. \end{cases} \tag{57}$$

Then, we can show that h_V solves (55) and (56) with $C = \{s \colon V(s) \le r\}$. So $h_* \le h_V$. But then $h_*(s) \le h_V(s) < 1$ if $s \notin C$, which shows that $R(s, s') < 1$ for $s \in C^c$ and $s' \in C$. Transience follows.

Example 4 cont'd Upper Hessenberg Transition Probabilities

Can we show that $\sum_n a_n > 1$ implies transience, using the above?

Example 8. Random Walks

We've already seen that the one-dimensional random walk is recurrent. To see that it's null recurrent, note that the transition probabilities are doubly stochastic. (Why?) So, $\rho = \rho P$ has a solution $\rho(i) \equiv 1$. This is not normalizable, so the chain is null recurrent.

Let's consider the case of a symmetric, two dimensional random walk which moves up, down, left, or right with probability 1/4 each. The chain is irreducible because we can find a non-zero probability path from any one state to another. It is also has period 2 for the same reason as in the one-dimensional case.

Thus, to assess the recurrence of the chain, we can look at only one point, let's say 0. We know, then, that $P^{2n+1}(0,0) = 0$ for all $n \geq 0$ and, by considering all paths two and from zero, we get that

$$P^{2n}(0,0) = \sum_{\substack{i,j \\ i+j=n}} \frac{(2n)!}{i!j!i!j!} \left(\frac{1}{4}\right)^{2n}. \tag{58}$$

Why?

Our goal is to compute $\sum_m P^m(0,0)$ to test recurrence for the chain. By multiplying $P^{2n}(0,0)$ by $(n!)^2/(n!)^2$, we get

$$\begin{aligned}
\sum_m P^m(0,0) &= \sum_n P^{2n}(0,0) && (59)\\
&= \sum_n \sum_{\substack{i,j \\ i+j=n}} \frac{(2n)!}{i!j!i!j!} \left(\frac{1}{4}\right)^{2n} && (60)\\
&= \sum_n \sum_{i=0}^{n} \frac{(2n)!}{i!(n-i)!i!(n-i)!} \left(\frac{1}{4}\right)^{2n} && (61)\\
&= \sum_n 4^{-2n} \binom{2n}{n} \sum_{i=0}^{n} \binom{n}{i}\binom{n}{i} && (62)\\
&= \sum_n 4^{-2n} \binom{2n}{n} \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} && (63)\\
&= \sum_n 4^{-2n} \binom{2n}{n}\binom{2n}{n} \qquad \text{(why?)} && (64)\\
&\sim \sum_n \frac{1}{\pi n} && (65)\\
&= \infty. && (66)
\end{aligned}$$

The penultimate equation follows by Stirling's approximation $n! \sim \sqrt{n} n^n e^{-n} \sqrt{2\pi}$, from which $\binom{2n}{n} \sim 2^{2n}/\sqrt{\pi n}$. Hence, the chain is recurrent, and because again the matrix is doubly stochastic, the invariant measure has infinite mass. The chain is thus positive recurrent.

Next, consider three dimensional case. Using the same logic, we have that

$$\begin{aligned}
\sum_m P^m(0,0) &= \sum_n P^{2n}(0,0) && (67)\\
&= \sum_n \sum_{\substack{i,j \\ 0 \leq i+j \leq n}} \frac{(2n)!}{i!j!(n-i-j)!i!j!(n-i-j)!} \left(\frac{1}{6}\right)^{2n} && (68)
\end{aligned}$$

$$= \sum_n 2^{-2n} \binom{2n}{n} \sum_{\substack{i,j \\ 0 \le i+j \le n}} \binom{n}{i\ j} \binom{n}{i\ j} 3^{-2n} \tag{69}$$

$$\le \sum_n 2^{-2n} \binom{2n}{n} c_n 3^{-n} \sum_{\substack{i,j \\ 0 \le i+j \le n}} \binom{n}{i\ j} 3^{-n} \tag{70}$$

$$= \sum_n 2^{-2n} \binom{2n}{n} c_n 3^{-n}, \tag{71}$$

where $c_n = \max_{\substack{i,j \\ 0 \le i+j \le n}} \binom{n}{i\ j}$. We can show that $c_n \sim \binom{n}{n/3\ n/3}$. We thus get that

$$P^{2n}(0,0) \le 2^{-2n} \binom{2n}{n} \frac{n!}{(n/3)!(n/3)!(n/3)!} 3^{-n} \sim C n^{-3/2}, \tag{72}$$

by Stirling's approximation. Thus, the chain is transient.

One more thing about these symmetric random walks in d dimensions. Consider a non-negative function V on the state space and let Δ be the drift operator $\Delta V = PV - V$ for the chain. When $d = 1$,

$$\Delta V(i) = \frac{1}{2}\left(V(i+1) + V(i-1)\right) - V(i) \tag{73}$$

$$= \frac{V(i+1) - 2V(i) + V(i-1)}{2} \tag{74}$$

$$= \frac{(V(i+1) - V(i)) - (V(i) - V(i-1))}{2}. \tag{75}$$

In general, we have that

$$\Delta V(i_1, \ldots, i_d) = \tag{76}$$

$$\frac{1}{d} \sum_{k=1}^{d} \frac{V(i_1, \ldots, i_{k-1}, i_k + 1, i_{k+1}, \ldots, i_d) - 2V(i_1, \ldots, i_d) + V(i_1, \ldots, i_{k-1}, i_k - 1, i_{k+1}, \ldots, i_d)}{2}, \tag{77}$$

which looks much worse than it is.

The operator Δ is a normalized (by $1/d$) version of an operator called the *discrete Laplacian* because on finer and finer grids, it approximates the standard Laplacian differential operator $\Delta = \nabla \cdot \nabla = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. Thus, the behavior of random walks on Euclidean space connects quite deeply to the solutions of differential equations.

For example, the existence of bounded functions V with ΔV satisfying certain inequalities corresponds to recurrence or transience of the chain. As another example, consider a region $S_0 \subset \mathcal{S}$ with boundary points ∂S_0. If f is a function on S_0 such that $\Delta f = 0$ on $S_0 - \partial S_0$ and $f = g$ on ∂S_0, then $f(s) = \mathsf{E}_s g(X_{T_{\partial S_0}})$.

So for example, in the one-dimensional symmetric random walk. Let $S_0 = \{0, \ldots, m\}$ and let $g(0) = 1$ and $g(m) = 0$. Then, the solution f gives the Gambler's ruin probabilities for every initial wealth. But solutions are just of the form

$$f(s) = 1 - \frac{s}{m} \tag{78}$$

by direct calculation.

We can even state a general theorem in this case.

Theorem 9. Let X be a symmetric random walk on $\mathcal{S} = \mathbb{Z}^d$. Let $S_0 \subset \mathcal{S}$ have boundary ∂S_0. Then, Δ restricted to S_0 corresponds to a symmetric matrix on S_0 and thus has eigenvalues (λ_k) and eigenfunctions (ϕ_k) and the solution to the above problem takes the form

$$f(s) = \frac{1}{d} \sum_k \left(\sum_{\substack{u \in S_0, v \in \partial S_0 \\ u, v \text{adjacent}}} \phi_k(u) g(v) \right) \frac{\phi_k(s)}{\lambda_k}. \tag{79}$$

Again, this looks bad, but don't worry about the details. The key point is that given the computable properties of the operator Δ and given boundary conditions, we can find what we need quite simply.

In the example above, we computed the solution by inspection because a linear function has zero second differences. But we can use the above theorem as well.

We can find by direct computation that

$$\phi_k(j) = \sqrt{\frac{2}{m}} \sin \frac{\pi j k}{m} \tag{80}$$

and $\lambda_k = 1 - \cos \pi k/m$. Then, using the above equation, we get

$$f(j) = \frac{1}{m} \sum_{k=1}^{m-1} \frac{\sin \frac{\pi k}{m} \sin \frac{\pi k j}{m}}{1 - \cos \frac{\pi k}{m}} = 1 - \frac{j}{m}, \tag{81}$$

as we found.

Example 10. Storage Model

Consider a storage system (dam, warehouse, insurance policy) that receives inputs at random times but otherwise drains at a regular rate.

Let $T_0 = 0$ and let the rest of the T_is be IID $\mathbb{Z}_\oplus$-valued with CDF G. These are inter-arrival times for the inputs to our storage system. Let the S_ns be IID $\mathbb{Z}_\oplus$ with CDF H. These are the amounts input at the time $Z_n = T_0 + \cdots + T_n$. Assume that the S_ns and T_ns are independent of each other as well. Suppose also that the storage system "drains" or outputs at rate r between inputs.

Define a process $(V_n)_{n\geq 0}$ by

$$V_{n+1} = (V_n + S_n - rT_{n+1})_+. \tag{82}$$

Here, V_n represents the contents of the storage system just before the nth input (that is, at time Z_n-).

Is V a Markov chain?

What is the structure of the transition probabilities?

What can we say about the long-run behavior of the chain?

What is special about the state $\{0\}$?

How might we generalize this model to make it more realistic?