Plan Martingales Results

- 1. Main Martingale Results
- 2. Some More Examples

Reading: G&S: 10.1–10.4 Next Time: Poisson and Renewal Processes

Homework 6 due Thursday, 6 April 2006.

Definitions 1.

A sequence of random variables $(K_n)_{n\geq 0}$ is called *predictable* (with respect to a filtration $(\mathcal{F}_n)_{n\geq 0}$) if K_n is \mathcal{F}_{n-1} -measurable for every $n\geq 1$.

A process $(K_n)_{n\geq 0}$ is called *increasing* if $\mathsf{P}\{K_n \leq K_{n+1}\} = 1$ for all $n \geq 0$.

Representation Theorem 2. The Doob Decomposition

A sub-martingale $Y = (Y_n)$ (with finite absolute expected values) can be decomposed *uniquely* (up to constant shifts) as

$$Y_n = M_n + K_n,\tag{1}$$

where $M = (M_n)$ is a martingale and $K = (K_n)$ is an increasing predictable process.

Proof.

Representation Theorem 3. If $X = (X_n)$ is a martingale, then we can write X as a partial sum as follows:

$$X_n = X_0 + \sum_{i=1}^n \Xi_i,$$
 (2)

where $\mathsf{E}(\Xi_{n+1} \mid \mathcal{F}_n) = 0.$

Proof.

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Idea 4. Upcrossings of a sequence.

Let $y = (y_n)_{n \ge 0}$ be a numeric sequence and let a < b. An *upcrossing* of the interval [a, b] by the sequence y is any pair i < j such $y_i \le a$, $a < y_k < b$ for i < k < j, and $y_j \ge b$.

Let $U_n(a, b, y)$ be the number of upcrossings in (y_0, \ldots, y_n) and $U(a, b, y) = \lim_{n \to \infty} U_n(a, b, y)$. What do these numbers tell us about the behavior of the sequence?

Lemma. Suppose that $U(a, b, y) < \infty$ for all rational a < b. Then, $\lim_{n\to\infty} y_n$ exists as an extended real number. (That is, the limit may be $\pm \infty$.)

Why? Let $l = \liminf_n y_n$ and $u = \limsup_n y_n$. Suppose the limit did not exist. Then, l < u and because the rationals are dense, we can find rationals a, b such that l < a < b < u. By definition of limit and limsup, both a and b are exceeded (below and above respectively) infinitely often. Thus, $U(a, b, y) = \infty$.

Thus, if we can bound $U(a, b, \cdot)$ for the sample paths of a stochastic process, we will get a limit.

Theorem 5. Upcrossings Inequality.

Suppose that $Y = (Y_n)$ is a submartingale and a < b. Than,

$$\mathsf{E}U_n(a,b,Y) \le \frac{\mathsf{E}(Y_n - a)_+ - \mathsf{E}(Y_0 - a)_+}{b - a}.$$
(3)

We will prove the upcrossings inequality later using other results, but for now, note what it implies about the regularity of the submartingale sample paths.

Theorem 6. The Martingale Convergence Theorem

Let $Y = (Y_n)$ be a submartingale (with respect to the filtration (\mathcal{F}_n)). If $\sup_{n>0} \mathsf{E} Y_n^+ < \infty$, then there exists a random variable Y_{∞} such that $\mathsf{P}\{Y_n \to Y_{\infty}\} = 1$.

In addition:

1. If $\mathsf{E}|Y_0| < \infty$, then $\mathsf{E}|Y_\infty| < \infty$.

2. If

$$\lim_{c \to \infty} \sup_{n \ge 0} \mathsf{E}\left(|Y_n| 1\{|Y_n| > c\}\right) = 0,\tag{4}$$

which we call *uniform integrability*, then

$$\lim_{n \to \infty} \mathsf{E}|X_n - X_\infty| = 0 \tag{5}$$

$$\mathsf{E}X_{\infty} = \mathsf{E}X_n \quad \text{for } n \ge 0. \tag{6}$$

What is the significance of the first condition in the theorem? When might that Question 7. fail? What does that uniform integrability condition mean? Can you find cases when it holds?

How does the upcrossings inequality help us sketch a proof?

Question 8. What can we say about a bounded submartingale?

Homework Question 9. Recall Doob's martingale. Let X be a random variable with $E|X| < \infty$. Define $Y_n = \mathsf{E}(X \mid \mathcal{F}_n)$ for a filtration (\mathcal{F}_n) .

Show that (Y_n) is a uniformly integrable sequence.

Intuitively, what form should Y_{∞} take?

Careful: don't ignore the filtration. Let \mathcal{F}_{∞} be the smallest σ -field containing every \mathcal{F}_n . This might be \mathcal{F} and might be smaller. Now, consider the question again.

Idea 10. Fun with Stopping Times

Stopping times play a big role in martingale theory. To see their importance, consider a gambling system that attempts to beat a fair game. What strategies do you have available? What information can you use to decide whether to stop playing? When you account for those constraints, can you beat the game?

That the answer is no is a fundamental result in the theory. Let's work up to it.

Proposition. Suppose $X = (X_n)$ is a submartingale with respect to (\mathcal{F}_n) and let T be a stopping time.

Then, the "stopped process" defined by $Y_n = X_{T \wedge n}$ is also a sub-martingale. Why?

So, stopping at a random time won't help our gambler. What about playing a different game? Suppose the gambler changes at some random time (necessarily a stopping time) and brings his current wealth to a new game.

Proposition. Let X and Y be martingales with respect to (\mathcal{F}_n) . Let T be a stopping time and suppose that $X_T = Y_T$ on the event $\{T < \infty\}$. Then, define

$$Z_n = X_n \mathbb{1}\{T > n\} + Y_n \mathbb{1}\{T \le n\}.$$
(7)

This too is a martingale with respect to (\mathcal{F}_n) .

Why is this true?

Now suppose our gambler wants to pick and choose which trials to bet on. This corresponds to finding a sequence of times on which to play.

Suppose $T_0 \leq T_1 \leq T_2 \leq \cdots$ is an increasing series of stopping times and X is a martingale. What can we say about the process $Y_n = X_{T_n}$ for $n \geq 0$.

Is it a martingale? In general, no. Consider the simple random walk with p = 1/2 and let T be the first hitting time of k > 0. Then, $\mathsf{E}S_0 = 0 \neq \mathsf{E}S_T = k$.

But the result is true if the stopping times are bounded.

Theorem 11. Optional Sampling

Let X be a submartingale with respect to (\mathcal{F}_n) .

- 1. If T is a stopping time for which $\mathsf{P}\{T \leq m\} = 1$ for some $m < \infty$, then $\mathsf{E}X_T^+ < \infty$ and $X_0 \leq \mathsf{E}(X_T \mid \mathcal{F}_0)$. (Hence, $\mathsf{E}X_0 \leq \mathsf{E}X_T$.)
- 2. If $T_0 \leq T_1 \leq \cdots$ are stopping times with $\mathsf{P}\{T_i \leq m_i\} = 1$ for $m_i < \infty$, then $Y_n = X_{T_n}$ is a submartingale as well.

Question 12. What happens in the above when X is a martingale? How do we prove this theorem?

Corollary 13. Optional Stopping Theorem Let X be a martingale and T be a stopping time such that

1. $P{T < \infty} = 1.$ 2. $E|X_T| < \infty$ 3. $EX_n 1{T > n} \rightarrow 0$ as $n \rightarrow \infty$.

Then, $\mathsf{E}X_T = \mathsf{E}X_0$.

Application 14. Wald's Identity.

Let (Ξ_n) be IID integrable random variables and N a stopping time of the sequence. What is $\mathsf{E}\sum_{i=1}^{N} \Xi_i$?

Applications for Next Time 15.

- 1. Option Pricing
- 2. False Discovery Rates

Example 16. Discretization and Derivatives

Let U be a Uniform (0,1) random variable. Define $X_n = k2^{-n}$ for the unique k such that $k2^{-n} \leq U < (k+1)2^{-n}$. As n increases, X_n gives finer and finer information about U.

Let f be a bounded function on [0, 1] and define

$$Y_n = 2^n \left(f(X_n + 2^{-n}) - f(X_n) \right).$$
(8)

What is Y_n approximating here as $n \to \infty$?

What is the distribution of U given X_0, \ldots, X_n ?

Show that Y_n is a martingale wrt X_n .