**Plan** Martingales Results and Applications

1. Main Martingale Results
2. Two selected Applications

Reading: G&S: 12.7, 12.8
*Next Time: Poisson and Renewal Processes*

Homework 7: G&S 12.3: 1–4; 12.4: 1, 2, 5, 6, 7; 12.5: 3,4; and question from class.
Due Thu 13 Apr 2006.

**Theorem 1.** The Martingale Convergence Theorem
Let $Y = (Y_n)$ be a submartingale (with respect to the filtration $(\mathcal{F}_n)$). If $\sup_{n \geq 0} \mathsf{E}Y_n^+ < \infty$, then there exists a random variable $Y_\infty$ such that $\mathsf{P}\{Y_n \to Y_\infty\} = 1$.
In addition:

1. If $\mathsf{E}|Y_0| < \infty$, then $\mathsf{E}|Y_\infty| < \infty$.
2. If
$$\lim_{c \to \infty} \sup_{n \geq 0} \mathsf{E}\left(|Y_n| 1\{|Y_n| > c\}\right) = 0, \tag{1}$$
which we call *uniform integrability*, then
$$\lim_{n \to \infty} \mathsf{E}|X_n - X_\infty| = 0 \tag{2}$$
$$\mathsf{E}X_\infty = \mathsf{E}X_n \quad \text{for } n \geq 0. \tag{3}$$

**Question 2.** What is the significance of the first condition in the theorem? When might that fail? What does that uniform integrability condition mean? Can you find cases when it holds?
How does the upcrossings inequality help us sketch a proof?

**Question 3.** What can we say about a bounded submartingale?

**Homework Question 4.** Recall Doob's martingale. Let $X$ be a random variable with $\mathsf{E}|X| < \infty$. Define $Y_n = \mathsf{E}(X \mid \mathcal{F}_n)$ for a filtration $(\mathcal{F}_n)$.

Show that $(Y_n)$ is a uniformly integrable sequence.

Intuitively, what form should $Y_\infty$ take?

Careful: don't ignore the filtration. Let $\mathcal{F}_\infty$ be the smallest $\sigma$-field containing every $\mathcal{F}_n$. This might be $\mathcal{F}$ and might be smaller. Now, consider the question again.

**Idea 5.** Fun with Stopping Times

Stopping times play a big role in martingale theory. To see their importance, consider a gambling system that attempts to beat a fair game. What strategies do you have available? What information can you use to decide whether to stop playing? When you account for those constraints, can you beat the game?

That the answer is no is a fundamental result in the theory. Let's work up to it.

*Proposition.* Suppose $X = (X_n)$ is a submartingale with respect to $(\mathcal{F}_n)$ and let $T$ be a stopping time.

Then, the "stopped process" defined by $Y_n = X_{T \wedge n}$ is also a sub-martingale.

Why?

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So, stopping at a random time won't help our gambler. What about playing a different game? Suppose the gambler changes at some random time (necessarily a stopping time) and brings his current wealth to a new game.

*Proposition.* Let $X$ and $Y$ be martingales with respect to $(\mathcal{F}_n)$. Let $T$ be a stopping time and suppose that $X_T = Y_T$ on the event $\{T < \infty\}$. Then, define

$$Z_n = X_n 1\{T > n\} + Y_n 1\{T \le n\}\,. \tag{4}$$

This too is a martingale with respect to $(\mathcal{F}_n)$.

Why is this true?

Now suppose our gambler wants to pick and choose which trials to bet on. This corresponds to finding a sequence of times on which to play.

Suppose $T_0 \le T_1 \le T_2 \le \cdots$ is an increasing series of stopping times and $X$ is a martingale. What can we say about the process $Y_n = X_{T_n}$ for $n \ge 0$.

Is it a martingale? In general, no. Consider the simple random walk with $p = 1/2$ and let $T$ be the first hitting time of $k > 0$. Then, $\mathsf{E}S_0 = 0 \ne \mathsf{E}S_T = k$.

But the result is true if the stopping times are bounded.

**Theorem 6.** Optional Sampling

Let $X$ be a submartingale with respect to $(\mathcal{F}_n)$.

1. If $T$ is a stopping time for which $\mathsf{P}\{T \le m\} = 1$ for some $m < \infty$, then $\mathsf{E}X_T^+ < \infty$ and $X_0 \le \mathsf{E}(X_T \mid \mathcal{F}_0)$. (Hence, $\mathsf{E}X_0 \le \mathsf{E}X_T$.)
2. If $T_0 \le T_1 \le \cdots$ are stopping times with $\mathsf{P}\{T_i \le m_i\} = 1$ for $m_i < \infty$, then $Y_n = X_{T_n}$ is a submartingale as well.

**Question 7.** What happens in the above when $X$ is a martingale?

How do we prove this theorem?

**Corollary 8.** Optional Stopping Theorem Let $X$ be a martingale and $T$ be a stopping time such that

1. $\mathsf{P}\{T < \infty\} = 1$.
2. $\mathsf{E}|X_T| < \infty$
3. $\mathsf{E}X_n 1\{T > n\} \to 0$ as $n \to \infty$.

Then, $\mathsf{E}X_T = \mathsf{E}X_0$.

**Application 9.** Wald's Identity.

Let $(\Xi_n)$ be IID integrable random variables and $N$ a stopping time of the sequence. What is $\mathsf{E}\sum_{i=1}^{N} \Xi_i$?

**Application 10.** False Discovery Rates
Suppose we have $m$ hypothesis tests and corresponding p-values $P_1, \ldots, P_m$. Let $P_{(0)} \equiv 0$ and $P_{(1)} < \cdots P_{(m)}$ are the p-values in increasing order. Then, Benjamini and Hocbherg (1995) used the following threshold to determine which null hypotheses would be rejected.

$$P^* = \max\left\{ P_{(i)} \colon P_{(i)} \le \alpha \frac{i}{m} \right\} \tag{5}$$

They showed that if we reject all null hypothesis for which $P_i \le P^*$, we will control the *false discovery rate* at level $\alpha$. That is,

$$\mathsf{E} \frac{\text{falsely rejected nulls}}{\text{rejected nulls}} \le \alpha. \tag{6}$$

Let's make this more precise.

Let $\mathcal{H}_0$ and $\mathcal{H}_1$ be a partition of $\{1, \ldots, m\}$. $\mathcal{H}_0$ represents the indices of true null hypotheses; $\mathcal{H}_1$ represents the indices of true alternatives (though which alternative is not specified by the index alone). Let $m_0 = \#(\mathcal{H}_0)$.

For any p-value *threshold*, $0 \le t \le 1$, define the following:

$$V(t) = \sum_{i \in \mathcal{H}_0} 1\{P_i \le t\} \tag{7}$$

$$R(t) = \sum_{i \in \mathcal{H}_0 \cup \mathcal{H}_1} 1\{P_i \le t\}. \tag{8}$$

$V(t)$ is the number of false rejects at threshold $t$; $R(t)$ is the total number of rejections. Define the False Discovery Proportion at threshold $t$ by

$$\mathrm{FDP}(t) = \frac{V(t)}{\max(R(t), 1)}. \tag{9}$$

Then, for a threshold $T$, fixed or random, the False Discovery Rate (FDR) is given by

$$\mathrm{FDR} = \mathsf{E}\,\mathrm{FDP}(T). \tag{10}$$

Benjamini and Hochberg (1995) showed that a threshold equivalent to

$$T_\alpha = \sup\left\{ t \in [0,1] \colon \frac{mt}{R(t)} \le \alpha \right\} \tag{11}$$

satisfies $\mathrm{FDR} \le \alpha$.

Let's give a martingale proof in three steps. But in this case we need to do two things: take the threshold as our time index, which is thus a continuous index, and reverse time. Both are technical variations on the discrete-time martingales we've seen, but the basic ideas are the same.

*Claim 1.* The process $t \mapsto V(t)/t$ is a reversed-time martingale with respect to the filtration

$$\mathcal{F}_t = \sigma\left(1\{P_i \le s\}, s \in [t, 1], i \in \{1, \ldots, m\}\right). \tag{12}$$

*Claim 2.* $T_\alpha$ is a stopping time (with time reversed) with respect to $(\mathcal{F}_t)$.

*Claim 3.* The optional stopping theorem implies that $\mathsf{E}\,\mathrm{FDP}(T_\alpha) = (m_0/m)\alpha$.

Proof of Claim 1. What do we need to show? What do we know (or need to assume)? What are the first steps?

Proof of Claim 2. What do we need to show? (It's not exactly the same as in the discrete case but what's the natural analogue?) What do we know (or need to assume)? How does $mt/R(t)$ change with $t$?

Proof of Claim 3. Thinking about the process $mt/R(t)$, we see that

$$R(T_\alpha) = \frac{m}{\alpha} T_\alpha. \tag{13}$$

Hence,

$$\mathrm{FDP}(T_\alpha) = \frac{\alpha}{m} \frac{V(T_\alpha)}{T_\alpha}. \tag{14}$$

Note that the stopped process is bounded by $m/\alpha$. Hmmmm...

The optional stopping theorem gives

$$\mathsf{E}\frac{V(T_\alpha)}{T_\alpha} = \mathsf{E}\frac{V(1)}{1} = m_0, \tag{15}$$

and thus

$$\mathrm{FDR}_{T_\alpha} = \frac{\alpha}{m}\mathsf{E}\frac{V(T_\alpha)}{T_\alpha} = \frac{m_0}{m}\alpha.$$

Sweet.

**Applications 11.** Options Pricing

Let $W_n$ be the price of an asset on day $n$. And let $Y_n$ be the ratio of the price on day $n$ to that on day $n-1$, so $W_n = wY_1 \cdots Y_n$ for current price $W_0 = w$.

Assume (for simplicity) a continuously-discounted inflation rate $\alpha \ge 0$, so that value at day $n$ is discounted by a factor $e^{-\alpha n}$.

Assume that $e^{-\alpha n} W_n$ is a martingale with respect to the $Y_n$s. Then, $\mathsf{E}e^{-\alpha n}W_n = \mathsf{E}W_0 = w$, and $\mathsf{E}W_n = we^{\alpha n}$ shows mean growth rate $\alpha$.

A call option contract enables one to purchase an asset at a fixed price regardless of the market price. Suppose for simplicity that the fixed price is 1 (we can always rescale). Then, if you held an option, you could exercise it whenever $W_n > 1$ for a profit of $W_n - 1$; when $W_n < 1$, there is no gain in exercising it. Hence, your profit from exercising the option at time $n$, suitably discounted would be $e^{-\alpha n}(W_n - 1)_+$.

How valuable is this option contract? If the value of the option grows at rate less than $\alpha$, then you might as well hold the asset instead.

Let $\beta \ge \alpha$ be the rate of return on the option, and assume that there exists a $q > 1$ such that

$$\mathsf{E}(Y_n^q \mid Y_1, \ldots, Y_{n-1}) \le e^{\beta}, \quad n = 1, 2, \ldots. \tag{16}$$

Define

$$f(w) = \begin{cases} \frac{w^q (q-1)^{q-1}}{q^q} & \text{if } w \le q/(q-1) \\ w - 1 & \text{if } w > q/(q-1). \end{cases} \tag{17}$$

We can show that $X_n = e^{-\beta n} f(W_n)$ forms a non-negative supermartingale. Hence, for any stopping time $T$,

$$f(w) = \mathsf{E}X_0 \ge \mathsf{E}e^{-\beta T} f(W_T). \tag{18}$$

Because $f(w) \ge (w-1)_+$, we have

$$\mathsf{E}e^{-\beta T}(W_T - 1)_+ \le f(w). \tag{19}$$

This gives us a bound on the discounted profit obtainable with any stopping strategy. In particular, if $w > q/(q-1)$, then the discounted value of the option is at most $w - 1$, suggesting that it be exercised immediately.

**Example 12.** Discretization and Derivatives

Let $U$ be a Uniform$\langle 0,1\rangle$ random variable. Define $X_n = k2^{-n}$ for the unique $k$ such that $k2^{-n} \le U < (k+1)2^{-n}$. As $n$ increases, $X_n$ gives finer and finer information about $U$.

Let $f$ be a bounded function on $[0,1]$ and define

$$Y_n = 2^n \left( f(X_n + 2^{-n}) - f(X_n) \right). \tag{20}$$

What is $Y_n$ approximating here as $n \to \infty$?

What is the distribution of $U$ given $X_0, \ldots, X_n$?

Show that $Y_n$ is a martingale wrt $X_n$.