Plan Renewal, Poisson, and Point Processes

1. More Renewal Process
2. The Poisson Process Revisited
3. Variants and Extensions
4. Point Processes

Reading: G&S: 7.1–7.3, 7.10
Next Time: Applications and Intro to Diffusions

Homework 8 due next Thursday

Reminder 1. The Basic Renewal Process
Let $X_1, X_2, \dots$ be a sequence of IID, non-negative random variables with common CDF F. Assume, to avoid odd boundary cases, that $F(0) = 0$. Let $\mu = \mathsf{E} X_1$.

We will interpret X_i as the time between the $i-1$st and ith event or "renewal".

Write $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$. Then, S_n is the time of the nth event.

Finally, define

$$N_t = \sup\{n \in \mathbb{Z}_\oplus : S_n \le t\}. \tag{1}$$

This counting process is called a *renewal process*. We showed last time that

$$N_t = \sum_{k=1}^\infty 1\{S_k \le t\}. \tag{2}$$

Definition 2. Let $m(t) = \mathsf{E} N_t$. This is called the *renewal function*. By the above representation, we have

$$m(t) = \sum_{k=1}^\infty \mathsf{P}\{S_k \le t\} = \sum_{k=1}^\infty F_k(t), \tag{3}$$

where F_k is the CDF of S_k.

Question 3. What is the distribution of S_k in terms of the distribution of S_{k-1}? (Start with $k = 1$.)

Derivation 4. The Distribution of N_t.

We have the following logical relation:

$$N_t \ge n \iff S_n \le t. \tag{4}$$

From this, we have

$$\begin{aligned} \mathsf{P}\{N_t = n\} &= \mathsf{P}\{N_t \ge n\} - \mathsf{P}\{N_t \ge n+1\} && (5)\\ &= \mathsf{P}\{S_n \le t\} - \mathsf{P}\{S_{n+1} \le t\} && (6)\\ &= F_n(t) - F_{n+1}(t). && (7) \end{aligned}$$

Notation 5. For our purposes here, we will use $\star$ for the convolution operator given by

$$(G \star H)(t) = \int_0^t G(t-s)\, dH(s), \tag{8}$$

whenever the integral exists, for real-valued functions G, H on $[0, \infty)$. You can show that this a commutative and associative operator.

Equation 6. A function u is a solution to a *renewal-type* equation if it satisfies

$$u = G + u \star F, \tag{9}$$

for bounded G on $[0, \infty)$.

In particular, the renewal function satisfies

$$m = F + m \star F. \tag{10}$$

There are a couple ways to see this. First, we can plug in $\sum_k F_k$ into the equation. Check. Second, we can condition on the first arrival. The function $\mathsf{E}(N_t \mid X_1 \text{ near } s)$ takes the following form:

$$\mathsf{E}(N_t \mid X_1 \text{ near } s) = \begin{cases} 1 + m(t-s) & \text{if } s \leq t \\ 0 & \text{if } s > t. \end{cases} \tag{11}$$

Why?

It follows that

$$\begin{aligned} m(t) &= \mathsf{E} N_t & (12)\\ &= \mathsf{E}\mathsf{E}(N_t \mid X_1) & (13)\\ &= \int_0^\infty \mathsf{E}(N_t \mid X_1 \text{ near } s)\, ds & (14)\\ &= \int_0^t (1 + m(t-s))\, dF(s) & (15)\\ &= F(t) + (m \star F)(t). & (16) \end{aligned}$$

More generally, a solution to the renewal-type equation (9) is given by

$$u = G + G \star m. \tag{17}$$

To see this, convolve both sides of the equation with F:

$$\begin{aligned} u \star F &= G \star F + G \star m \star F & (18)\\ &= G \star F + G \star (m - F) & (19)\\ &= G \star m & (20)\\ &= u - G, & (21) \end{aligned}$$

hence,

$$u = G + u \star F. \tag{22}$$

We can also show that if G is bounded on compact sets, then u is a unique solution and is also bounded on compact sets.

Definition 7. A real-valued random variable is said to be *lattice* if it takes values in a set $\{ka\colon k \in \mathbb{Z}\}$. The *span* (or period) of the lattice is the maximal such a.

Limit Theorems 8. Renewal processes behave asymptotically as you might expect. (ha ha) As $t \to \infty$, the following hold

1. $\dfrac{N_t}{t} \to \dfrac{1}{\mu}$ almost surely.
2. $\dfrac{m(t)}{t} \to \dfrac{1}{\mu}$.
3. $\dfrac{m(t+h) - m(t)}{h} \to \dfrac{1}{\mu}$, for all h if X_1 is not lattice and for all h that are integer multiples of the span if X_1 is lattice.

In addition, N_t/t, properly centered and scaled, is asymptotically Normal. See the book (section 10.2) for proofs.

Process 9. The *Poisson Process* is a renewal process with interarrival distribution F equal to an Exponential$\langle\lambda\rangle$. More precisely, this is called a (homogeneous) *Poisson process* with rate λ.

Recall that from this setup, we showed that the Poisson process has two other important properties.

First, *independent increments*. The random variables $N_t - N_s$ and $N_v - N_u$ are independent whenever $(s, t]$ and $(u, v]$ are disjoint. (The same goes for multiple such intervals.)

Second, *stationary increments*, the distribution of $N_t - N_s$ depends only on $t - s$.

We can thus characterize this process in two other equivalent ways.

Alternate Definition 1 10.
Let $N = (N_t)_{t\geq 0}$ be a process with the following properties:

1. $N_0 = 0$
2. For $0 \leq s < t$, $N_t - N_s$ has a Poisson$\langle\lambda(t - s)\rangle$ distribution.
3. For $0 \leq s_1 < t_1 < s_2 < t_2 < \cdots < s_m < t_m$, the random variables $N_{t_i} - N_{s_i}$ are independent.

Then, N is a Poisson process with rate λ.

Alternate Definition 2 11.
Let $N = (N_t)_{t\geq 0}$ be a process with the following properties:

1. $N_0 = 0$
2. The process has stationary and independent increments.
3. $\mathsf{P}\{N_h = 1\} = \lambda h + o(h)$.
4. $\mathsf{P}\{N_h \geq 2\} = o(h)$.

Then, N is a Poisson process with rate λ.

Theorem 12. The two alternate definitions are equivalent.

Basic idea of the proof: Poisson approximation to the binomial.

Note that from these alternative definitions, we can derive the interarrival distribution by

$$\mathsf{P}\{X_1 > t\} = \mathsf{P}\{N_t = 0\} = e^{-\lambda t}. \tag{23}$$

Useful Fact 13. ... which you derived in homework.

Given $N_t = n$, the set of n arrival times $\{S_1, \ldots, S_n\}$ have the same distribution as the order statistics of n independent Uniform$\langle 0, t\rangle$ random variables.

Process 14. By generalizing the alternate definitions, we get a whole new class of interesting processes.

Let $N = (N_t)_{t \geq 0}$ be a process with the following properties:

1. $N_0 = 0$
2. The process has independent increments.
3. $\mathsf{P}\{N_{t+h} - N_h \geq 2\} = o(h)$.
4. $\mathsf{P}\{N_{t+h} - N_h = 1\} = \lambda(t)h + o(h)$.

Then, N is called an inhomogeneous Poisson process with rate function $\lambda(t)$.

We can show that $N_{t+s} - N_t$ has a Poisson distribution with mean $m(t+s) - m(t)$, where $m(t) = \int_0^t \lambda(s)\, ds$.

Perspective 15. A different view of the Poisson process.

We think of the Poisson process as generated from the random scatter of points on $[0, \infty)$. The process N then counts the points in any set. That is, for a (measurable) set A, $N(A)$ is a random variable counting the number of points in A. N becomes a *random measure*.

To formalize this, consider the collections of random variables $N(A)$ for all (Borel) sets A, where $N(A)$ counts the number of "points" in the set A.

Let Λ be a measure on $[0, \infty)$ such that $\Lambda(A) < \infty$ for every bounded set A. And assume that for every collection of disjoint, bounded Borel sets $A_1, \ldots, A_k$, we have

$$\mathsf{P}\{N(A_j) = n_j, j = 1, \ldots, k\} = \prod_{j=1}^{k} \frac{(\Lambda(A_j))_j^n}{n_j!} e^{-\Lambda(A_j)}. \tag{24}$$

That is, the $N(A_j)$s are independent Poisson$\langle\Lambda(A_j)\rangle$ random variables.

Let's see how this works. Suppose that $\Lambda(A) = \lambda\ell(A)$, where $\ell(A)$ is the Lebesgue measure ("length") of A and $\lambda > 0$. Then, we have that

1. $N(\{0\}) = 0$
2. The probability above shows that $N(s_j, t_j]$ are independent for disjoint intervals $s_j < t_j$.
   It also shows that for any bounded Borel sets $A_1, \ldots, A_k$, the joint distribution of the random variables $N(A_1 + t), \ldots, N(A_k + t)$ does not depend on t, where $A + t$ represents $\{a + t : a \in A\}$, for all t such that all of the shifted sets are contained in $[0, \infty)$.
   In particular, $N(s, t]$ has the same distribution as $N(s + h, t + h]$ for all $h \geq -s$.
   Thus, we have stationary and independent increments.
3. $\mathsf{P}\{N(0, h] = 1\} = \lambda h e^{-\lambda h} = \lambda h + o(h)$.
4. $\mathsf{P}\{N(0, h] > 1\} = \sum_{k=2}^{\infty} \frac{(\lambda h)^k}{k!} e^{-\lambda h} = o(h)$.

This gives us back the homogeneous Poisson process.

Now suppose that λ is a non-negative function, and define $\Lambda(A) = \int_A \lambda(x)\, dx$. Then, we have

1. $N(\{0\}) = 0$
2. Independent increments from the probability above.
3. $\mathsf{P}\{N(t, t + h] = 1\} = \int_t^{t+h} \lambda(x)\, dx e^{-\lambda h} = \lambda(t)h + o(h)$.
4. $\mathsf{P}\{N(t, t + h] > 1\} = \sum_{k=2}^{\infty} \frac{(\int_t^{t+h} \lambda(x)\, dx)^k}{k!} e^{-\lambda h} = o(h)$.

And we get the inhomogeneous Poisson process.

Definition 16. A *counting measure* ν on a space $\mathcal{S}$ is a measure with the following additional properties:

1. $\nu(A)$ is non-negative-integer valued for any measurable A.
2. $\nu(A) < \infty$ for any bounded, measurable A.

Any counting measure ν on $\mathcal{S}$ has the form

$$\nu = \sum_i k_i \delta_{x_i}, \tag{25}$$

for a countable collection of positive integers k_i and points $x_i \in \mathcal{S}$, where δ_x is a point-mass at x. If all the $k_i = 1$, then ν is said to be a *simple* counting measure.

Definition 17. A *point process* is a random counting measure.

That is, if point process is a (measurable) mapping of each $\omega \in \Omega$ to a counting measure.

Example 18. The homogeneous and inhomogeneous Poisson processes.

Example 19. Let M be a non-negative integer-valued random variable with distribution G. Given $M = m$, draw $X_1, \ldots, X_m$ be IID from F. Define

$$N(A) = \sum_{i=1}^{M} 1\{X_i \in A\}. \tag{26}$$

Then, $N(A)$ is a point process.

Example 20. Let M be a non-negative integer-valued random variable with distribution G. Let μ be a probability measure on a space $\mathcal{S}$.

Given any $k-1$ disjoint sets $A_1, \ldots, A_{k-1}$, let $A_k = (\overset{k-1}{\underset{1}{\cup}} A_j)^c$. Let

$$\mathsf{P}\{N(A_j) = n_j, j = 1, \ldots, k\} = \binom{M}{n_1, \ldots, n_k} \mu^{n_1}(A_1) \cdots \mu^{n_k}(A_k). \tag{27}$$

What is the marginal distribution of $N(A_1)$ here? Note that $M = N(\mathcal{S})$.

We care not just about the distributions of $N(A)$ but also moments. For example, if A_1 and A_2 partition $\mathcal{S}$:

$$\mathsf{E}(N(A_1)N(A_2) \mid M) = M(M-1)\mu(A_1)\mu(A_2). \tag{28}$$

So $\mathsf{Cov}(N(A_1), N(A_2)) = c_{[2]}\mu(A_1)\mu(A_2)$, where $c_{[2]}$ is second factorial cumulant of M.

These distributions and moments are easiest to study with generating functions.