Plan Group Work

1. Brownian Motion
2. Point Processes

Reading: G&S 13.4–13.5
Next Time: Information Theory.

Homework 9 due next Friday. Do 8 of the problems below, at least 5 of which must come from the Brownian motion section. I highly recommend 2, 3, 5, 6, 7, 8, and 9, but you can choose as you like.

Brownian Motion

Reminder. Brownian Motion, aka The Weiner Process

A *Weiner process* (or *Brownian motion*) $W = (W_t)_{t \geq 0}$ is a real-valued Gaussian process satisfying:

1. W has independent and stationary increments
2. $W_{t+h} - W_t$ has a Normal$\langle 0, \sigma^2 h \rangle$ distribution, for all $t, h \geq 0$ and a constant $\sigma^2 > 0$.
3. The sample paths of W are almost surely continuous.

A *standard Weiner process* (or *standard Brownian motion*) has $W_0 = 0$ and $\sigma^2 = 1$.

Among the interesting properties of W:

1. W is a Gaussian process with mean function $\mu(t) = 0$ and autocovariance $\rho(s,t) = \min(s,t)$.
2. W is a Markov process.
3. W is a Martingale.
4. Extends easily to multiple dimensions.
5. The sample paths of W are nowhere differentiable.
6. In a sense we discussed, W is integrated white noise.

Below, let $W = (W_t)_{t \geq 0}$ denote a standard Weiner process unless otherwise noted

Question 1. A Gaussian process X is a Markov process if and only if

$$\mathsf{E}(X_{t_n} \mid X_{t_{n-1}} \text{ near } u_{n-1}, \ldots, X_{t_1} \text{ near } u_1) = \mathsf{E}(X_{t_n} \mid X_{t_{n-1}} \text{ near } u_{n-1}), \tag{1}$$

for all $u_1, \ldots, u_{n-1}$ and all times $t_1 < \ldots < t_n$.

Show that a standard Weiner process W is a Markov process using this definition.

Question 2. Show that a standard Weiner process is a martingale. In addition, show that both

$$U_t = W_t^2 - t \tag{2}$$

$$V_t = \exp\left(\lambda W_t - \frac{1}{2}\lambda^2 t\right), \quad \lambda \in \mathbb{R}, \tag{3}$$

are martingales with respect to the filtration $\mathcal{F}_t = \sigma(W_s, s \leq t)$.

Question 3. Let W be a standard Weiner process. Let $a < 0 < b$ and let T be the first hitting time of $\{a, b\}$. That is,

$$T = \inf\{t \geq 0 \colon W_t \in \{a, b\}\}. \tag{4}$$

Assume that you can apply analogues of the optional sampling/stopping theorem in the discrete case.

Using the martingales in the previous question, show that $\mathsf{E}T < \infty$, find the probability that W hits b before a, and compute $\mathsf{E}T$.

Hint: Don't try to find $\mathsf{E}T$ until the end; for the first step, only a bound is needed. Consider stopping the process at $T \wedge n$.

Question 4. Let X be a real-valued, continuous-time Markov process such that

$$\mathsf{P}\{X_{t+h} \text{ near } v \mid X_t \text{ near } u\} = \pi(v, h \mid u)\, dv, \tag{5}$$

for all $t, h \geq 0$ and $u, v \in \mathbb{R}$.

Suppose that $g(u, t)$ satisfies

$$g(u, h) = \int \pi(v, s \mid u) g(v, t+s)\, dv, \tag{6}$$

for $s, t > 0$.

Let $Z_t = g(X_t, t)$ and assume that $\mathsf{E}|Z_t| < \infty$ for all t. Show that $Z = (Z_t)_{t \geq 0}$ is a martingale with respect to the filtration $\mathcal{F}_t = \sigma(X_s, s \leq t)$.

Show that $g(x, t) = x^2 - t$ and $g(x, t) = e^{\lambda x - \frac{1}{2}\lambda^2 t}$ satisfy equation (6) with $\pi(v, s \mid u)$ a Normal$\langle u, t \rangle$ density.

Question 5. Let $B_t = W_t - tW_1$ for $0 \leq t \leq 1$.

Show that this is a Gaussian process and find it's mean and autocovariance function. Heuristically/intuitively, what can you say about this process?

Question 6. Our goal in this question is to show that for every $t > 0$,

$$\lim_{n \to \infty} \sum_{k=1}^{2^n} \left(W_{k2^{-n}t} - W_{(k-1)2^{-n}t} \right)^2 = t, \tag{7}$$

where the convergence takes place almost surely and in $\mathcal{L}^2$.

The quantity in the equation is called the quadratic variation of the process. If we naively think of the sum as a typical Reimann sum, this limit would produce the integral relationship

$$\int_0^t (dW_s)^2 = t = \int_0^t ds, \tag{8}$$

or put another way:

$$(dW_s)^2 = ds. \tag{9}$$

Whoa! (Note the connection to the $(\Delta x)^2/(\Delta t) = D$ in the construction from last time.) This very odd equation can be given rigorous meaning. "dW" plays the role of white noise, and we get the "multiplication table"

$$dt \cdot dt = 0 \tag{10}$$

$$dt \cdot dW_t = dW_t \cdot dt = 0 \tag{11}$$

$$dW_t \cdot dW_t = dt. \tag{12}$$

But I digress.

First, show that equation (7) implies that

$$\lim_{n \to \infty} \sum_{k=1}^{2^n} \left| W_{k2^{-n}t} - W_{(k-1)2^{-n}t} \right| = \infty, \tag{13}$$

which means that the Brownian motion sample paths have infinite total variation on every compact interval. This does not prove but is consistent with the statement that the sample paths are nowhere differentiable.

Second, prove equation (7). To do this, it is helpful to introduce some auxilliary variables:

$$\Delta_{nk} = W_{k2^{-n}t} - W_{(k-1)2^{-n}t} \tag{14}$$

$$U_{nk} = \Delta_{nk}^2 - t2^{-n}. \tag{15}$$

Re-express the equation in terms of the U_{nk}s. What do you know about the joint distribution of the U_{nk}s?

HINT: If Z has a Normal$\langle 0, \sigma^2 \rangle$ distribution, $\mathsf{E}Z^4 = 3\sigma^4$.

Show convergence in $\mathcal{L}^2$ (i.e., mean square).

Next, use Chebychev's inequality to bound $\mathsf{P}\left\{\left|\sum_{k=1}^{2^n} U_{nk}\right| > \epsilon\right\}$. Use the Borel-Cantelli lemma to show the probability that $\left|\sum_{k=1}^{2^n} U_{nk}\right| > \epsilon$ infinitely often is zero. Show how to get almost sure convergence as a result.

Question 7. Because sample paths of the Weiner process are continuous, we can make use of the "reflection principle." This question illustrates the use of that principle. Somewhat hand-wavy but very useful.

Fix a constant $c > 0$. Consider a collection of sample paths of W over an interval $0 \le s \le T$ such that $W_T > c$. By continuity, every path in this collection must take the value c for some $0 \le s \le T$. Let u denote the first hitting time of c along each path. Note that $u = u(\omega)$ depends on the sample path.

Next, *reflect* each path about the horizontal line at c to get a collection of sample paths

$$V_s = \begin{cases} W_s & \text{if } s \le u \\ 2c - W_s & \text{if } s > u \end{cases}. \tag{16}$$

(Remember there are a set of such paths.) Draw a picture of one path in the first set and its reflection, to help fix the ideas.

The distribution of the path given $W_u = c$ is symmetric in that a set of paths and the set of their reflections have the same probability. How would you argue this?

Thus, we have two collections of sample paths (the original and their reflections), both of which have their maximum over $0 \le s \le T$ bigger than c and both of which have the same probability. Using the fact that $\{W_T = c\}$ has probability 0, find a nice expression for $\mathsf{P}\{\max_{0\le s\le T} W_s \ge c\}$.

Using the reflection principle, it is also possible to prove that the probability that W has at least one zero in the interval (a, b) is

$$\frac{2}{\pi} \arccos \sqrt{\frac{a}{b}}.$$

You don't have to prove that, but it might be fun to try sometime.

Point Processes

Question 8. Suppose that Pittsburgh city buses arrive a bus stop every $\Delta > 0$ minutes and that a passanger arrives at the bus stop at some random point uniformly distributed over the day. How long on average does the passenger wait?

Suppose instead that the arrival times of the busses at the stop are described by a Poisson random measure on the line with mean measure $\nu = \text{Leb}/\Delta$. How long on average does the passenger wait?

What's going on here?

Question 9. Suppose the positions of trees in a forest are well characterized by a Poisson random measure N with mean measure $\nu = \lambda\text{Leb}$ for $\lambda > 0$. What is the distribution of the distance from an arbitrary point in the forest to the nearest tree? Suppose that each tree has a small radius $a > 0$. What is the distance from the arbitrary point that you can see to the east?

Question 10. A point process N is *stationary* if for all bounded, measurable sets $A_1, \ldots, A_k$ for $k \geq 1$, the joint distribution of $N(A_1 + t), \ldots, N(A_k + t)$ does not depend on t over all t such that the shifted sets $A_i + t$ remain in the domain of N.

Show that the homogenous Poisson process is stationary in this sense.

Question 11. Let N be a point process with mean measure ν, and define

$$C_2(A, B) = \mathsf{Cov}(N(A), N(B)). \tag{17}$$

This is called the *covariance measure* of the processs. What does this mean in a sentence or two? Can we simplify this in the stationary case?

Find the covariance measure for a Poisson random measure.

Question 12. Let N be a point process. Write

$$V(A) = \mathsf{P}\{N(A) = 0\}. \tag{18}$$

This is called the *avoidance measure* of the process.

Find the avoidance measure for the Poisson random measure.

Question 13. The K-function $K(t)$ for a stationary point process is defined as the expected number of points within a distance t of a typical point, divided by the overall intensity of the process.

Show that for a homogeneous Poisson scatter in d dimensions, $K(t) = v_d t^d$, where v_d is the volume of the unit ball in d dimensions.