1. First return to zero for a simple random walk

1.1. From last time: 31 Jan

Let $T = \min\{n \geq 1 \text{ such that } S_n = 0\}$.

Let $u_n = \mathsf{P}\{S_n = 0\}$ and $f_n = \mathsf{P}\{T = n\} = \mathsf{P}\{S_1 \neq 0, \ldots, S_{n-1} \neq 0, S_n = 0\}$.

We know that $u_0 = 1$ and $f_0 = 0$. For $n \geq 1$, condition on T:

$$\begin{aligned}
u_n &= \mathsf{P}\{S_n = 0\} \\
&= \mathsf{E}\mathsf{P}\{S_n = 0 \mid T\} \\
&= \sum_{k=1}^{n} \mathsf{P}\{S_n = 0 \mid T = k\}\, \mathsf{P}\{T = k\} \\
&= \sum_{k=1}^{n} u_{n-k} f_k \\
&= \sum_{k=0}^{n} u_{n-k} f_k \qquad [\text{because } f_0 = 0]
\end{aligned}$$

Constructing the generating functions gives:

$$\begin{aligned}
U(z) &= 1 + \sum_{n \geq 0} \sum_{k=0}^{n} u_{n-k} f_k z^n \\
&= 1 + \sum_{k \geq 0} f_k z^k \sum_{n \geq k} u_{n-k} z^{n-k} \\
&= 1 + G_T(z) U(z)
\end{aligned}$$

Hence,

$$G_T(z) = 1 - \frac{1}{U(z)}.$$

Note that $u_n = 0$ if n odd. If $n = 2m$ is even, $S_n = 0$ iff there are as many upward as downward steps. Hence,

$$u_{2m} = (pq)^m \binom{2m}{m},$$

and

$$\begin{aligned}
U(z) &= \sum_{m=0}^{\infty} \binom{2m}{m} (pq)^m z^{2m} \\
&= \sum_{m=0}^{\infty} \binom{2m}{m} (pqz^2)^m \\
&= \frac{1}{\sqrt{1 - 4pqz^2}}.
\end{aligned}$$

Note

$$\binom{2m}{m} = (-1)^m 4^m \binom{-1/2}{m}$$

and use the binomial theorem.

Hence, $G_T(z) = 1 - \sqrt{1 - 4pqz^2}$.

Note, if we take $G_T(1)$, we get $1 - \sqrt{1-4pq}$, where $4pq = 4p - 4p^2$. So, $1 - 4pq = (2p-1)^2$ Then,

$$\begin{aligned} G_T(1) &= 1 - |2p - 1| \\ &= 1 - |p - (1-p)| \\ &= 1 - |p - q|. \end{aligned}$$

It eventually returns if $p = q$!

Now, take $p = q = 1/2$ and compute $G_T'(z)$. We get $G_T'(1) = \infty$ in this case, so $\mathsf{E}T = \infty$.

1.2. From Today: 2 Feb

Note that by the binomial theorem

$$\sqrt{1+z} = \sum_{k \geq 0} \binom{\frac{1}{2}}{k} z^k = 1 + \sum_{k \geq 1} \binom{2(k-1)}{k-1} \frac{(-1)^{k-1}}{k 2^{2k-1}} z^k.$$

Hence,

$$\begin{aligned} F(z) &= 1 - 1 - \sum_{k \geq 1} \binom{2(k-1)}{k-1} \frac{(-1)^{k-1}}{k 2^{2k-1}} (-4)^k (pq)^k z^{2k}. \\ &= \sum_{k \geq 1} \binom{2(k-1)}{k-1} \frac{2}{k} (pq)^k z^{2k} \\ &= \sum_{k \geq 1} \binom{2k}{k} \frac{(pq)^k}{2k-1} z^{2k}. \end{aligned}$$

because

$$\frac{2}{k}\binom{2(k-1)}{k-1} = \frac{2k(2k-1)}{k^2(2k-1)}\binom{2(k-1)}{k-1} = \binom{2k}{k}\frac{1}{2k-1},$$

by the absorption identity. So, $\mathsf{P}\{T = 2k\} = 0$ for $k = 0$ and for $k \geq 1$,

$$\mathsf{P}\{T = 2k\} = \binom{2k}{k} \frac{p^k q^k}{2k-1} = \frac{\mathsf{P}\{S_{2k} = 0\}}{2k-1}.$$

Or put another way,

$$\mathsf{P}\{T = n\} = \frac{1}{n-1} \mathsf{P}\{S_n = 0\} \, 1_{\mathbb{Z}_+}(n)$$

2. Hitting time

Same basic analysis as with the return time but a different conditioning setp.

Let $T_r = \min\{n \geq 1 \text{ such that } S_n = r\}$ be the first hitting time of $r \geq 1$.

Let $f_{r,n} = \mathsf{P}\{T_r = n\} = \mathsf{P}\{S_1 \neq r, \ldots, S_{n-1} \neq r, S_n = r\}$.

Condition on T_1:

$$\begin{aligned}
\mathsf{P}\{T_r = n\} &= \mathsf{E}\mathsf{P}\{T_r = n \mid T_1\} \\
&= \sum_{k=1}^{\infty} \mathsf{P}\{T_r = n \mid T_1 = k\} \, \mathsf{P}\{T_1 = k\} \\
&= \sum_{k=1}^{n-1} \mathsf{P}\{T_r = n \mid T_1 = k\} \, \mathsf{P}\{T_1 = k\} \\
&= \sum_{k=1}^{n-1} f_{r-1,n-k} f_{1,k}
\end{aligned}$$

Thus, $G_{T_r}(z) = G_{T_{r-1}}(z) G_{T_1}(z)$ which implies by induction that

$$G_{T_r}(z) = \left(G_{T_1}(z)\right)^r .$$

What does this imply about the distribution of T_r?

Now, note that $f_{1,0} = 0$, $f_{1,1} = p$, and

$$f_{1,n} = p\mathsf{P}\{T_1 = n \mid X_1 = 1\} + q\mathsf{P}\{T_1 = n \mid X_1 = -1\} = p1_{(n=1)} + qf_{2,n-1}.$$

Expanding the recurrence gives

$$G_{T_1}(z) = zp + zqG_{T_2}(z) = zp + zqG^2_{T_1}(z).$$

So, $zqG^2_{T_1}(z) - G_{T_1}(z) + zp = 0$. There are two solutions to this quadratic equation, but only one of them is a G because we require that $G_{T_1}(0) = 0$ (take limits).

Thus, we get the unique solution:

$$G_{T_1}(z) = \frac{1 - \sqrt{1 - 4pqz^2}}{2qz}.$$

Using the above analysis, we get

$$\begin{aligned}
G_{T_1}(z) &= \frac{1}{2qz} \sum_{k \geq 1} \binom{2(k-1)}{k-1} \frac{2}{k} (pq)^k z^{2k} \\
&= \sum_{k \geq 1} \binom{2(k-1)}{k-1} \frac{2k-1}{k} \frac{1}{2k-1} \, p^k q^{k-1} z^{2k-1} \\
&= \sum_{k \geq 1} \binom{2k-1}{k} \frac{1}{2k-1} \, p^k q^{k-1} z^{2k-1}.
\end{aligned}$$

Hence, for $k \geq 1$,

$$\mathsf{P}\{T_1 = 2k-1\} = \frac{\mathsf{P}\{S_{2k-1} = 1\}}{2k-1}.$$

Or re-expressed:

$$\mathsf{P}\{T_1 = n\} = \frac{1}{n}\,\frac{\mathsf{P}\{S_n = 1\}}{1}_{(n>0,\, n \text{ odd})}.$$

But what about $G_{T_r}(z)$? We can guess from the above what the form will be. Is it true?
Lemma. For $r > 0$,

$$\left(\frac{1-\sqrt{1-4z}}{2z}\right)^r = \sum_{k=0}^{\infty} \frac{r}{k+r}\binom{2k+r-1}{k} z^k.$$

Proof by Lagrange Inversion Formula below.

Using the lemma, we get for $r > 1$,

$$\begin{aligned}
G_{T_r}(z) &= (G_{T_1}(z))^r \\
&= \left(\frac{1-\sqrt{1-4pqz^2}}{2qz}\right)^r \\
&= p^r z^r \left(\frac{1-\sqrt{1-4pqz^2}}{2pqz^2}\right)^r \\
&= p^r z^r \sum_{k=0}^{\infty} \frac{r}{k+r}\binom{2k+r-1}{k} p^k q^k z^{2k} \\
&= \sum_{k=0}^{\infty} \frac{r}{k+r}\binom{2k+r-1}{k} p^{k+r} q^k z^{2k+r} \\
&= \sum_{k=0}^{\infty} \frac{r}{k+r}\binom{2k+r-1}{k+r-1} p^{k+r} q^k z^{2k+r} \\
&= \sum_{k=0}^{\infty} \frac{r}{2k+r}\binom{2k+r}{k+r} p^{k+r} q^k z^{2k+r}
\end{aligned}$$

Hence,

$$\mathsf{P}\{T_r = 2k+r\} = \frac{r}{2k+r}\binom{2k+r}{k+r} p^{k+r} q^k = \frac{r}{2k+r}\mathsf{P}\{S_{2k+r} = r\}.$$

So, for $r \geq 1$.

$$\mathsf{P}\{T_r = n\} = \frac{r}{n}\mathsf{P}\{S_n = r\}.$$

For $r < 0$, the situation is reversed with p and q exchanging roles. But this is just the expression $\mathsf{P}\{S_n = r\}$ again. Thus, for $r \neq 0$,

$$\mathsf{P}\{T_r = n\} = \frac{|r|}{n}\mathsf{P}\{S_n = r\}\, 1_{\mathbb{Z}_+}(n).$$

Nice.

Now, proving the lemma leads us to the Lagrange Inversion Formula stated in Theorem 9 in the handout.

Let's prove the lemma in a related form. We want

$$[z^n]\left(\frac{\sqrt{1+4z}-1}{2}\right)^k$$

for each n and k. Let $F(u) = u^k$. Note that $\frac{\sqrt{1+4z}-1}{2}$ is the root of the polynomial equation $u^2 + u - z = 0$. Solving this gives

$$u = \frac{z}{1+u}.$$

Aha! Take $G(u) = 1/(1+u)$. Now apply the theorem

$$\begin{aligned}
[z^n]F(U(z)) &= [z^n]U^k(z) \\
&= \frac{1}{n}[u^{n-1}](F'(u)G^n(u)) \\
&= \frac{1}{n}[u^{n-1}]\frac{ku^{k-1}}{(1+u)^n} \\
&= \frac{1}{n}[u^{n-1}]ku^{k-1}\sum_{j\geq 0}\binom{-n}{j}u^j \\
&= \frac{k}{n}[u^{n-1}]\sum_{j\geq 0}(-1)^j\binom{n+j-1}{j}u^{k+j-1} \\
&= \frac{k}{n}(-1)^{n-k}\binom{2n-k-1}{n-k}.
\end{aligned}$$

Very Cool! The transformation to what we need in the lemma is straightforward.