Pentagon Walk

Let $X_0, X_1, \ldots$ be random variables that take values in the set $\{A, B, C, D, E\}$. Suppose X_n is the node being visited at time n in the Pentagon walk we discussed in class. That is, we assume that $X_0 = A$ with probability 1 and that X_n moves to each node adjacent to node X_{n-1} with probability 1/2, independently of where it was before.

Let $A_n = \mathsf{P}\{X_n = A\}$ and similarly for B_n, C_n, D_n, and E_n. We have

$$A_n = 1_{(n=0)} + \frac{1}{2}B_{n-1} + \frac{1}{2}E_{n-1}, \tag{1}$$

where all these terms are 0 for $n < 0$. What are the corresponding equations for the other nodes?

Let $S_A(z) = \sum_n A_n z^n$ and similarly for the other nodes. Multiplying both sides of these recurrences by z^n and summing over all integers, we get

$$\begin{aligned}
\mathsf{S}_A(z) &= 1 + \mathsf{S}_B(z)\,\frac{z}{2} + \mathsf{S}_E(z)\,\frac{z}{2} \\
\mathsf{S}_B(z) &= \qquad \mathsf{S}_A(z)\,\frac{z}{2} + \mathsf{S}_C(z)\,\frac{z}{2} \\
\mathsf{S}_C(z) &= \qquad \mathsf{S}_B(z)\,\frac{z}{2} + \mathsf{S}_D(z)\,\frac{z}{2} \\
\mathsf{S}_D(z) &= \qquad \mathsf{S}_C(z)\,\frac{z}{2} + \mathsf{S}_E(z)\,\frac{z}{2} \\
\mathsf{S}_E(z) &= \qquad \mathsf{S}_A(z)\,\frac{z}{2} + \mathsf{S}_D(z)\,\frac{z}{2}
\end{aligned}$$

This corresponds to the following linear system:

$$\begin{pmatrix} 1 & -\frac{z}{2} & 0 & 0 & -\frac{z}{2} \\ -\frac{z}{2} & 1 & -\frac{z}{2} & 0 & 0 \\ 0 & -\frac{z}{2} & 1 & -\frac{z}{2} & 0 \\ 0 & 0 & -\frac{z}{2} & 1 & -\frac{z}{2} \\ -\frac{z}{2} & 0 & 0 & -\frac{z}{2} & 1 \end{pmatrix} \begin{pmatrix} \mathsf{S}_A(z) \\ \mathsf{S}_B(z) \\ \mathsf{S}_C(z) \\ \mathsf{S}_D(z) \\ \mathsf{S}_E(z) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving this system by one of many means gives us:

$$\begin{aligned}
\mathsf{S}_A(z) &= \frac{4 - 2\,z - z^2}{4 - 2\,z - 3\,z^2 + z^3} \\
\mathsf{S}_B(z) &= \frac{z\,(2 - z)}{4 - 2\,z - 3\,z^2 + z^3}
\end{aligned}$$

$$
\begin{aligned}
\mathsf{S}_C(z) &= \frac{z^2}{4 - 2\,z - 3\,z^2 + z^3} \\
\mathsf{S}_D(z) &= \frac{z^2}{4 - 2\,z - 3\,z^2 + z^3} \\
\mathsf{S}_E(z) &= \frac{z\,(2 - z)}{4 - 2\,z - 3\,z^2 + z^3}
\end{aligned}
$$

Now,

$$
\begin{aligned}
S_A(z) &= \frac{\left(1 + \frac{z}{2\phi}\right)\left(1 + \frac{z}{2\widehat{\phi}}\right)}{(1 - z)\left(1 - \frac{z}{2\phi}\right)\left(1 - \frac{z}{2\widehat{\phi}}\right)} \\
&= \frac{u_1}{1 - z} + \frac{u_2}{1 - \frac{z}{2\phi}} + \frac{u_3}{1 - \frac{z}{2\widehat{\phi}}}.
\end{aligned}
$$

Expanding this out, we get

$$
\begin{pmatrix} 1 & 1 & 1 \\ \frac{1}{2} & -(1 - \frac{1}{2\widehat{\phi}}) & -(1 + \frac{1}{2\phi}) \\ -\frac{1}{4} & \frac{1}{2\widehat{\phi}} & \frac{1}{2\phi} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{4} \end{pmatrix}.
$$

We can solve this directly to get

$$
u_1 = \frac{1}{5} \qquad u_2 = \frac{2}{5} \qquad u_3 = \frac{2}{5}.
$$

Thus,

$$
\begin{aligned}
S_A(z) &= \frac{1}{5}\left[\frac{1}{1 - z} + \frac{2}{1 - \frac{z}{2\phi}} + \frac{2}{1 - \frac{z}{2\widehat{\phi}}}\right] \\
&= \frac{1}{5}\left[\frac{1}{1 - z} + \frac{2}{1 + \frac{\widehat{\phi} z}{2}} + \frac{2}{1 + \frac{\phi z}{2}}\right] \\
&= \frac{1}{5}\sum_{n=0}^{\infty}\left[1 + \frac{(-1)^n}{2^{n-1}}(\phi^n + \widehat{\phi}^n)\right] z^n.
\end{aligned}
$$

Note that $\phi^n + \widehat{\phi}^n = F_n + 2F_{n-1} = F_{n+1} + F_{n-1}$, where F_n is the nth Fibonacci number.

So,

$$A_n = \mathsf{P}\{X_n = A\} = \frac{1}{5}\left(1 + \frac{(-1)^n}{2^{n-1}}(\phi^n + \widehat{\phi}^n)\right).$$

This converges to $1/5$ as $n \to \infty$.

A few early values:

$$\mathsf{S}_A(z) = 1 + \frac{z^2}{2} + \frac{3\,z^4}{8} + \frac{z^5}{16} + \frac{5\,z^6}{16} + \frac{7\,z^7}{64} + \frac{35\,z^8}{128} + \cdots \tag{2}$$