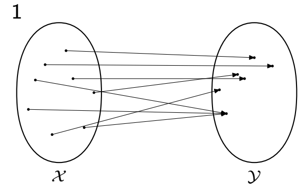
F

FUNCTIONS

1. Fundamentals

Suppose that we have two non-empty sets $\mathcal{X}$ and $\mathcal{Y}$ (keep in mind that these two sets may be equal). A *function* that maps $\mathcal{X}$ to $\mathcal{Y}$ is rule that associates to each element $x \in \mathcal{X}$ one and only one element $y \in \mathcal{Y}$. (Functions are sometimes called mappings to emphasize this association between the two sets.) The figure[1] shows a "conceptual icon" to illustrate this idea. As usual, we like to give objects names. For example, we might say that f is a function. To indicate that f maps $\mathcal{X}$ to $\mathcal{Y}$ we write[2] $f\colon \mathcal{X} \to \mathcal{Y}$. When speaking, we read "$f\colon \mathcal{X} \to \mathcal{Y}$" as "f maps $\mathcal{X}$ to $\mathcal{Y}$" or "f from $\mathcal{X}$ to $\mathcal{Y}$" depending on the context.

1



2 If $\mathcal{X}$ and $\mathcal{Y}$ are clear from context, we need not state the $:\mathcal{X} \to \mathcal{Y}$ part.

You have seen functions before. In your math courses, you were probably asked to do something like the following: "Graph the curve $y = x^2$." What you were representing with your graph was a function because to each number on the "x-axis" you associated one and only one number on the "y-axis". (This would fail, for instance, if graph folded over on itself.) The y-coordinate is computed in this case by squaring the x-coordinate, which gives a definite rule for mapping one set of numbers (the real numbers) to another set of numbers (also the real numbers). That's all a function is.

Let's call this function f. What do we know about it? First, f maps real numbers to real numbers, so we can write $f\colon \mathbb{R} \to \mathbb{R}$. Moreover, we know that to any real number x, f associates the real number x^2. We write this in argument-result form as $f(x) = x^2$. The value of x is an argument to the function and the value x^2 is the result.[3]

3 In C++ notation, we would write this function as

```
real f( real x )
{
    return( x*x );
}
```

which emphasizes the distinction between argument and return value.

There are a few things to notice about this function f. First, different values of the argument can give the same return value, for instance $x = \pm 1$ both give $f(x) = 1$, but each value of the argument only has one associated return value. This uniqueness property is at the heart of the definition of a function. Second, the argument can be any real number whatsoever, but the result is always non-negative. Thus it would be just as correct to write $f\colon \mathbb{R} \to [0, \infty[$. Which is to be preferred is largely cosmetic; unless we have a good reason we go with the option that is simpler to write.[4] Note that it is not accurate to write $f\colon \mathbb{R} \to [0, 1[$ since $f(2) = 4$ is not an element of $[0, 1[$.

4 See the discussion of domain, codomain, and range below.

A quick aside on the terminology. The phrase "Graph the curve $y = x^2$" from your algebra and calculus classes is a bit too loose for our purposes. We will express this in two steps. First, we define the function: "Let f be the function $f(x) = x^2$." What we mean by this is that $f\colon \mathbb{R} \to \mathbb{R}$ is the function that takes x as an argument and returns x^2 for any $x \in \mathbb{R}$. It is important to note that the variable x here is a *local variable*; it is a placeholder for defining the function that has no meaning outside the definition. The variable `x` in the code of note 3 above has the same property; the compiler would not recognize as the same variable any reference to `x` outside the function. Second, having defined f we can "Graph the function f". This emphasizes that the object of interest is the function itself and deemphasizes the role played by dummy variables like y and x. Incidently, the graph itself is one way of identifying the function; it is useful for numerically valued functions like $f(x) = x^2$. Sometimes thinking of the two sets separately with the function represented by explicit mappings between points (as in the figures above) can be more helpful. Pictures can be a big help in thinking about mathematical ideas like functions. Remember in either case that, in general, functions are mappings from one set to another.

Thought Question Draw a curve on the x-y plane that does not represent the graph of a function.

Thought Question Can the graph of a valid function have breaks or jumps in it? Why or why not?

Thought Question Define a function that maps $\mathbb{Z}_+$ to $\mathbb{Z}$ so that every $k \in \mathbb{Z}$ is the value of the function for some $j \in \mathbb{Z}_+$.

Let $f: \mathcal{X} \to \mathcal{Y}$ be a general function. If $x \in \mathcal{X}$, we say that f "takes the value" or "maps to" $f(x)$ at x. The set $\mathcal{X}$ is called the *domain* of the function f and the set $\mathcal{Y}$ is called the *codomain* of the function f. The domain is the set of possible arguments of f, and the codomain is the set of possible values f can take. Often, as in the case of $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$, the set of values that a function takes must be a subset of but need not be equal to the codomain. The set $\{f(x) \text{ such that } x \in \mathcal{X}\}$ is called the *range* of f.

The choice of codomain for a function is largely arbitrary as long as the chosen set contains the range of the function. Unless there is some compelling reason to choose a more specific set, in practice, we usually define functions so that the codomain is some familiar space, such as $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_\oplus$, $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{Z}_\oplus$, $\mathbb{C}$, or $\mathbb{R}^n$ for some $n > 1$. In this case, where the codomain of a function g is a named set $\mathcal{Y}$, we say that g is $\mathcal{Y}$-valued.[5] The common cases are as follows for a function g:

g takes values in	g is a ____ function
$\mathbb{R}$	real-valued
$\mathbb{R}_+$	positive real-valued
$\mathbb{R}_\oplus$	non-negative real-valued
$\mathbb{Z}$	integer-valued
$\mathbb{Z}_+$	positive integer-valued
$\mathbb{Z}_\oplus$	non-negative integer-valued
$\mathbb{Q}$	rational-valued
$\mathbb{C}$	complex-valued
$\mathbb{R}^n$	$\mathbb{R}^n$-valued

Notice that these categories are not necessarily exclusive. For example, an integer-valued function takes values in $\mathbb{R}$ since $\mathbb{Z} \subset \mathbb{R}$, so it is also a real-valued function. There is another, related distinction to be made. If g is $\mathbb{R}^n$-valued for integer $n > 1$, then we say that g is *vector valued*.[6] Otherwise, if g takes values in some subset of $\mathbb{R}$ or $\mathbb{C}$, we say that g is *scalar valued*.

Comparing Functions. Functions are mathematical objects in their own right. We can name them, operate on them, study sets or sequences of them. We can also compare them. Two functions are equal if they have the same domain and if for every element of the domain the two

5 When we want to be precise, we say that $g: \mathcal{X} \to \mathcal{Y}$ is a $\mathcal{Y}$-valued function defined on $\mathcal{X}$.

6 See subsection 2.7 for more details about vector-valued functions.

functions take the same value.

> **Conditions for Equality of Functions.**
> Two functions f and g are equal if and only if they have the same domain, and
>
> $$f(x) = g(x) \text{ for all } x \text{ in their common domain.} \quad \text{(F.1)}$$

Because real numbers can be ordered – there is a way of determining which of two numbers is bigger – some real-valued functions can also be ordered. Consider the functions $f(x) = e^x$ and $g(x) = 1 + x$ defined on $\mathbb{R}$. No matter which $x \in \mathbb{R}$ you consider it is always true that $f(x) \geq g(x)$. It makes sense, then, to consider the function f to be $\geq$ the function g.

> **Conditions for Ordering Real-Valued Functions.**
> For functions f and g defined on the same domain and taking values in $\mathbb{R}$, we have $f \leq g$ if and only if
>
> $$f(x) \leq g(x) \text{ for all } x \text{ in their common domain.} \quad \text{(F.2)}$$
>
> The same conditions apply with $>$, $<$, or $\geq$ replacing $\leq$.

Thus, for example, a non-negative (positive) real-valued function f satisfies $f \geq 0$ ($f > 0$), where 0 is the constant function on the domain of f,

Function Images. If $\mathcal{A} \subset \mathcal{X}$, then the *image* of $\mathcal{A}$ is the subset of the codomain $\mathcal{Y}$ given by

$$f(\mathcal{A}) = \{f(x) \text{ such that } x \in \mathcal{A}\}.$$

The image of $\mathcal{A}$ tells us the set of all results we obtain by giving every element of $\mathcal{A}$ as a function of f. For $f(x) = x^2$, $f(\{-1, 1\}) = \{1\}$. To generate, a more complicated example, look at the image of a set of points in $\mathcal{X}$ in figure 1. The range of f is just the set $f(\mathcal{X})$ which is necessarily a subset of $\mathcal{Y}$.

If $\mathcal{B} \subset \mathcal{Y}$, then the *inverse image* of $\mathcal{B}$ is the subset of the domain $\mathcal{X}$ given by

$$f^{-1}(\mathcal{B}) = \{x \in \mathcal{X} \text{ such that } f(x) \in \mathcal{B}\}.$$

The inverse image of $\mathcal{B}$ tells us all the possible arguments whose values lie in $\mathcal{B}$. For $f(x) = x^2$, $f^{-1}(\{1\}) = \{-1, 1\}$. To generate, a more complicated example, look at the inverse image of a set of points in $\mathcal{Y}$ in figure 1.

Thought Question If $f\colon \mathcal{X} \to \mathcal{Y}$ is a function $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{X}$, is $f(\mathcal{A}_1 \cap \mathcal{A}_2) = f(\mathcal{A}_1) \cap f(\mathcal{A}_2)$? If so, why, and if not, given an example where it fails.

Thought Question If $f\colon \mathcal{X} \to \mathcal{Y}$ is a function, what is $f^{-1}(\emptyset)$? What is $f^{-1}(\mathcal{Y})$?

Thought Question If $f(x) = x^2$, what is $f^{-1}(\{-1, 1\})$?

Thought Question If $f\colon \mathcal{X} \to \mathcal{Y}$ is a function, $\mathcal{A} \subset \mathcal{X}$, and $\mathcal{B} \subset \mathcal{Y}$, then explain with a couple sentences, with a picture, or with an example why $f(f^{-1}(\mathcal{B})) \subset \mathcal{B}$ and $f^{-1}(f(\mathcal{A})) \supset \mathcal{A}$.

2. Some Important Examples

2.1. Constant Functions

The simplest function is a *constant function* which takes the same value for every argument. For example, $f(x) = 1$, $g(x) = 0$, and $h(x) = 117.017394$ for $x \in \mathbb{R}$ are all real-valued constants. A general function $f\colon \mathcal{X} \to \mathcal{Y}$ is constant if there is a $y \in \mathcal{Y}$ such that $f(x) = y$ for all $x \in \mathcal{X}$.

In practice, we treat the notation for constant functions a bit loosely, by allowing a constant value c to stand for the number and the function. For example, we use the symbol 1 for both the number 1 and the function that takes the value 1 for every point in its domain. Which meaning is being used should be clear from context. Thus, when we write $\mathsf{E}1 = 1$, the 1 on the left is a function because the expected value operator acts on functions and the 1 on the right is a number because the expected value operator returns a number.

2.2. Indicator Functions

An indicator function is any function that take only the values 0 and 1. These will be very important in our work.

If $\mathcal{A}$ is a subset of $\mathcal{X}$ for any set $\mathcal{X}$, then the indicator function $1_{\mathcal{A}}: \mathcal{X} \to \mathbb{R}$ is defined by

$$1_{\mathcal{A}}(x) = \begin{cases} 1 & \text{if } x \in \mathcal{A} \\ 0 & \text{if } x \notin \mathcal{A}. \end{cases} \tag{F.3}$$

The notation "1" is intended as a mnemonic; this function takes the value 1 on the set given in the subscript and is zero otherwise. We call this the "indicator of $\mathcal{A}$".

Thought Question To get a feel for indicators, sketch the graph of the indicators of the following sets: $[0,1]$, $[-2,-1] \cup [1,2]$, $[0,\infty[$, and $\mathbb{R}$.

Thought Question Describe the function $1_{\mathcal{X}}: \mathcal{X} \to \mathbb{R}$ more simply.

Thought Question If $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$, define $1_{\mathcal{A}\cap\mathcal{B}}$ using $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$.

Thought Question If $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$ (i.e., $\mathcal{A}$ and $\mathcal{B}$ are disjoint), define $1_{\mathcal{A}\cup\mathcal{B}}$ in terms of $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$.

Thought Question Define the function $1_{\mathrm{compl}(\mathcal{A})}$ in terms of $1_{\mathcal{A}}$.

Indicators are very useful for representing functions that take on a finite number of values. For example, suppose the function g is defined as follows:

$$g(x) = \begin{cases} 3 & \text{if } x > 4 \\ -2 & \text{if } 2 \le x \le 4 \\ 1 & \text{if } -1 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

Then, we can write

$$g = 3 \cdot 1_{]4,\infty[} - 2 \cdot 1_{[2,4]} + 1_{]-1,2[}\,.$$

To see this, plug in a variety of arguments on both sides and compare the values. Notice that only one of the indicators in the above expression is ever non-zero. In general, suppose g is a function that takes the value y_i on the set $\mathcal{A}_i$ for $i = 1, \ldots, n$, where $y_1, \ldots, y_n \in R$ and the sets $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are disjoint. We could write this g as

$$g(x) = \begin{cases} y_1 & \text{if } x \in \mathcal{A}_1 \\ y_2 & \text{if } x \in \mathcal{A}_2 \\ \ldots & \text{if } \ldots, \end{cases}$$

but this is both tedious and can make g hard to work with. Instead, we can write

$$g = \sum_{i=1}^{n} y_i \cdot 1_{\mathcal{A}_i}.$$

2.3. Polynomials

If x is a real number, then powers of x, such as $x \cdot x$, $x \cdot x \cdot x$, $x \cdot x \cdot x \cdot x$, and so forth, are also real numbers, which we denote respectively by x^2, x^3, x^4, and so forth. Notice that because multiplication is commutative $x^2 x^4 = x^6$; more generally for integers j, k, $x^k x^j = x^{k+j}$ for the same reason. Incidentally, by this rule, $x^4 x^{-1} = x^3$;. But what does x^{-1} mean? In order to cancel the extra factor of x, we must take $x^{-1} = 1/x$. It follows by similar logic that $x^{-k} = 1/x^k$ for any positive integer. Similarly, $x^0 = 1$ because $0+0+0+\cdots = 1$. As suggested by the above equation, we define $x^{-1} = 1/x$ Thus, for each $k \in \mathbb{Z}$, we define the kth degree *power function* which maps x to x^k. If k is even, then the power function x^k takes only non-negative values; if k is odd, then the power function x^k takes all real values. Whenever $k \neq 0$, the power function x^k takes the value 0 at 0. The function x^0 is just the constant 1.

Suppose that $a_0, a_1, \ldots, a_m$ are real numbers. Then, the function that maps x to $a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$ is called an *mth degree polynomial.* A basic theorem of mathematics states that every mth degree polynomial has exactly m *complex* points at which the polynomial is zero.[7] There may be fewer than m such points in $\mathbb{R}$, however. Another important theorem states that any sufficiently smooth function can be approximated arbitrarily well by a polynomial of some (potentially large) degree.

7 These are called roots of the polynomial.

2.4. Exponential Functions and Logarithms

One of the fundamental functions of mathematics is the exponential function, which maps x to e^x, where e is the irrational number $2.71828182846\ldots$.[8] This function arises in so many different contexts in mathematics that can arguably be called the most important mathematical function. Here we give a few useful facts about the exponential

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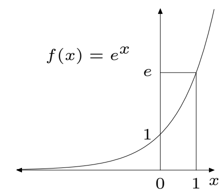

9 By the properties listed above, notice that $e^{x\log(b)} = e^{\log(b^x)} = b^x$.

function:

Properties of the Exponential Function

For any $x, u \in \mathbb{R}$,

1. $e^x > 0$, $e^0 = 1$, and $e^1 = e$.
2. $e^x \to \infty$ and $e^{-x} \to 0$, as $x \to \infty$.
3. $e^{x+u} = e^x\, e^u$.
4. $$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$
5. $\frac{d}{dx}\, e^x = e^x$ and $\int_{-\infty}^{x} e^t\, dt = e^x$.

Notice that from the above properties e^x increases as x increases, from 0 "at" $-\infty$ to ∞ "at" ∞. Suppose that $e^{x_1} = e^{x_2}$, then by property 3 above, $1 = e^{x_1}/e^{x_2} = e^{x_1}e^{-x_2} = e^{x_1 - x_2}$ which implies that $x_1 - x_2 = 0$, or $x_1 = x_2$. In other words, given any $y > 0$, we can find some $x \in \mathbb{R}$ with $e^x = y$. That value x is called the *logarithm* of y, denoted by $\log y$. Whereas e^x maps $\mathbb{R}$ onto $]0, \infty[$, log maps $]0, \infty[$ onto $\mathbb{R}$. Here we give a few useful facts about the logarithm function:

Properties of the Logarithm Function

For any $y, v > 0$ and $p \in \mathbb{R}$,

1. $e^{\log(y)} = y$ and $\log(e^x) = x$.
2. If $0 < y < 1$, $\log(y) < 0$; $\log(1) = 0$; and if $y > 1$, $\log(y) > 0$.
3. As $y \to \infty$, $\log(y) \to \infty$ and as $y \to 0$, $\log(y) \to -\infty$.
4. $\log(y\, v) = \log(y) + \log(v)$.
5. $\log(1/y) = -\log(y)$.
6. $\log(y^p) = p \log(y)$.
7. $\frac{d}{dy} \log(y) = 1/y$ and $\int_0^y \log(t), dt = y\log(y) - y$.

We defined the logarithm of y as the number x such that $e^x = y$; we call this the logarithm "base e". We could just as easily define a logarithm "base 10" (the number x such that $10^x = y$) or logarithm "base 2" (the number x such that $2^x = y$) or logarithm "base b" for any $b > 0$.[9]

Because of the importance of the exponential function, the logarithm base e is called the *natural logarithm.* If we want to talk about other bases, we will denote it with a subscript on the "log", as in $\log_{10}$, $\log_2$, and $\log_b$.

2.5. Trigonometric Functions

The trigonometric functions sin and cos are intimately related to the exponential function through the relationship between complex numbers:

$$e^{\imath\theta} = \cos\theta + \imath\sin\theta.$$

As x changes, $e^{\imath x}$ moves around the unit circle in the complex plane.[10] What this tells us is that the coordinates of a point on the unit circle an angle θ from the positive x-axis are $(\cos\theta, \sin\theta)$.[11] This observation yields the key identity: $\cos^2\theta + \sin^2\theta = 1$. The functions cos and sin satisfy a huge variety of useful identities, including the following:

10 Taking $\theta = \pi$ yields the miraculous relationship

$$e^{\imath\pi} + 1 = 0$$

among the five most important numbers in mathematics.

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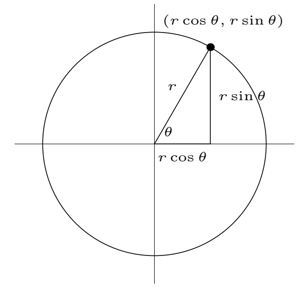

Trigonometric Identities
For any $\theta, \phi \in \mathbb{R}$,

1. $\cos^2\theta + \sin^2\theta = 1$,
2. $\cos\theta = \sin(\pi/2 - \theta)$ and $\sin\theta = \cos(\pi/2 - \theta)$,
3. $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$ and
   $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$,
4. $\sin' = \cos$ and $\cos' = -\sin$.

From sin and cos, we can produce the other trigonometric functions including $\tan = \sin/\cos$, $\cot = \cos/\sin$, $\sec = 1/\cos$, $\csc = 1/\sin$. For example, $\tan(\theta)$ is defined for $-\pi/2 < \theta < \pi/2$ and satisfies identities like $\sec^2\theta - \tan^2\theta = 1$.

Finally, over limited ranges these trigonometric functions are invertible. The inverses are usually obtained by putting the word "arc" in front of the name. Thus, we have arcsin, arccos, arctan, and so forth.

2.6. Other Commonly Used Functions

The *Gamma Function* $\Gamma(x)$ arises frequently in probabilistic calculations. It is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t}\, dt.$$

While this may look daunting, you are actually already familiar with the Gamma Function: For an integer k,

$$\Gamma(k+1) = k! = k(k-1)\ldots 1, \tag{F.4}$$

where $k!$ is read “k factorial”. Hence, $\Gamma(1) = \Gamma(2) = 1$, $\Gamma(4) = 6$, and so forth. This follows from the more general recurrence relationship which the Γ satisfies:

$$\Gamma(x+1) = x\,\Gamma(x).$$

We can also compute that $\Gamma(1/2) = \sqrt{\pi}$, so it follows that $\Gamma(3/2) = (1/2)\sqrt{\pi}$, $\Gamma(5/2) = (3/4)\sqrt{\pi}$, and so forth.

In general, there is no explicit expression for $\Gamma(x)$ in terms of elementary functions when x is not an integer or half integer. A formula called Stirling’s Approximation does give us a useful approximation to Γ for large x, and hence to $n!$ for large n:

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{\sqrt{2\pi}\, x^{x+1/2}\, e^{-x}} = 1. \tag{F.5}$$

In other words, if n is large, we can approximate $n!$ by $\sqrt{2\pi}\, n^{n+1/2}\, e^{-n}$.

The *floor* and *ceiling* functions map a real number x to a nearby integer. The floor of x, denoted by $\lfloor x \rfloor$ is the largest integer that is less than or equal to x. For example, $\lfloor 1.5 \rfloor = 1 = \lfloor 1 \rfloor$ and $\lfloor -0.25 \rfloor = -1$. The ceiling of x, denoted by $\lceil x \rceil$, is the smallest integer that is greater than or equal to x. For example, $\lceil 1.5 \rceil = 2 = \lceil 2 \rceil$ and $\lceil -0.25 \rceil = 0$. Notice that if k is an integer, then $k = \lfloor k \rfloor = \lceil k \rceil$.

2.7. Vector Functions

2.7.1. $\mathbb{R}^n$-valued functions

A vector is, in essence, an indexed list of numbers. A vector valued function can therefore be viewed as an indexed list of functions; this view is helpful in working with such functions.

If, for some $n > 1$, g is an $\mathbb{R}^n$-valued function defined on a set $\mathcal{X}$, then we can write

$$g(x) \equiv (g_1(x), g_2(x), \ldots, g_n(x)), \tag{F.6}$$

where the g_i's are real-valued functions defined on $\mathcal{X}$. These are referred to as the *component functions* of g. Therefore, to specify an $\mathbb{R}^n$-valued function, we need only give its n component functions, and we can move freely between working with g or the g_is.

Most of the operators we use work transparently on vector-valued functions. For example, if we can define an integral $\int$ and derivative d/dx operators on $\mathcal{X}$, then we have

$$\begin{aligned}\int g(x)\,dx &= (\int g_1(x)\,dx, \ldots, \int g_n(x)\,dx)\\ \frac{d}{dx}g(x) &= (\frac{d}{dx}g_1(x), \ldots, \frac{d}{dx}g_n(x)).\end{aligned}$$

That is, the operator just acts on each component, giving a new vector.

More importantly, the same story holds for the expected value and distribution operators. If $Y = (Y_1, \ldots, Y_n)$ is any $\mathbb{R}^n$-valued random variable and if X is any random variable (scalar or vector valued), we have that

$$\begin{aligned}\mathsf{E}Y &= (\mathsf{E}Y_1, \ldots, \mathsf{E}Y_n)\\ \mathsf{D}_X g &= \mathsf{E}g(X) = (\mathsf{E}g_1(X), \ldots, \mathsf{E}g_n(X)).\end{aligned}$$

2.7.2. Real-valued functions defined on $\mathbb{R}^n$

We can also considered real-valued functions defined on $\mathbb{R}^n$ for some $n > 1$. If h is such a function and $x = (x_1, \ldots, x_n)$ is an n vector in $\mathbb{R}^n$ with components $x_1, \ldots, x_n \in \mathbb{R}$, then we can write the value of h at x in two ways:

$$h(x) \;\equiv\; h(x_1, \ldots, x_n). \tag{F.7}$$

We treat these as completely equivalent and interchangeable.

When integrating functions defined on $\mathbb{R}^n$, we use the analogous equivalence for infinitesimals

$$dx \equiv dx_1\, dx_2\, \cdots\, dx_n. \tag{F.8}$$

Thus, in $\mathbb{R}^2$, dx is an infinitesimal area element and in $\mathbb{R}^3$, dx is an infinitesimal volume element. These directly generalize the infinitesimal length element in $\mathbb{R}$.

Using these forms for the infinitesimal, we can write the integral of h in either of two equivalent ways

$$\int h(x)\, dx \;\equiv\; \int\int \cdots \int h(x_1, \ldots, x_n)\, dx_1\, dx_2\, \cdots\, dx_n. \tag{F.9}$$

The former is usually more convenient when working with integrals in the abstract. The operation of integrating a function like h has the same structure regardless of the dimension of the dimension. The latter is more convenient when actually doing the calculations.

2.7.3. Summary

Vector Function Equivalences

- A vector-valued function g can be defined in terms of real-valued component functions $g_1, \ldots, g_n$ as in equation (F.6).
- The operators E, D_X, as well as integration and differentiation, operate on vector-valued functions to produce a vector by acting on the real-valued component functions.
- A real-valued function h defined on $\mathbb{R}^n$ can be written with its argument in vector or component form as in equation (F.7).
- The integral of a real-valued function h defined on $\mathbb{R}^n$ can be written in either vector or component form as in equation (F.9).

3. How to make new functions from old ones

Most of the functions we will deal with are either in the list of examples above or are constructed from other functions by one of the methods described below.

Composition. If $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y} \to \mathcal{Z}$ are functions, then the *composite function* $g \circ f\colon \mathcal{X} \to \mathcal{Z}$ defined by

$$(g \circ f)(x) = g(f(x)).$$

This function can only be defined if the range of f is a subset of the domain of g.

> **Thought Question** If f and g are as above, can $g \circ f$ and $f \circ g$ be different functions? Can they be the same function? Give examples.

Linear Combinations. Consider the power functions x^0, x, x^2, and x^3. If I have constants 1.2, 3.1, -3, and 10, I can form a new function $f(x) = 1.2x^0 + 3.1x - 3x^2 + 10x^3$, which is just a cubic polynomial. The operation of multiplying several function by constants and then adding the results together is called taking a *linear combination* of the functions. Suppose we have functions $f_1, \ldots, f_m$ with the same domain $\mathcal{X}$ and a codomain that is either $\mathbb{R}^k$ (or $\mathbb{C}^k$) for some $k \in \mathbb{Z}_+$. Given real (or complex) constants $a_1, \ldots, a_m$, we can form a linear combination of the f_i's

$$f = a_1 f_1 + \cdots + a_m f_m,$$

which is defined by

$$f(x) = a_1 f_1(x) + \cdots + a_m f_m(x), \quad x \in \mathcal{X}.$$

As you may have guessed, polynomials are just linear combinations of power functions.

Products. If I have two functions $f, g\colon \mathcal{X} \to \mathbb{C}$,[12] then the product $f \cdot g$ is a new function defined by

$$(f \cdot g)(x) = f(x)g(x), \quad x \in \mathcal{X}.$$

Notice that since all constants can be considered functions on any domain, we can always multiply a function by a constant to get a new function.

12 Any functions $\mathcal{X} \to \mathbb{R}$ satisfy this condition.

Cumulative Integrals. If $f: \mathbb{R} \to \mathbb{R}$, then we can define

$$F(x) = \int_{-\infty}^{x} f(t)\, dt.$$

Such a *cumulative integral function* has at least one derivative, and $F'(x) = f(x)$. Moreover, if $f(t) \geq 0$ for every $t \in \mathbb{R}$, $F(x)$ cannot decrease as x gets larger; that is, $y \geq x$ implies that $F(y) \geq F(x)$. To see this, think about the area under a curve accumulating as x increases; if the curve is always above zero, then the area is always increasing.

Thought Question Draw the graph of the cumulative integral of $1_{[0,1]}$.

Derivatives. If $f: \mathbb{R} \to \mathbb{R}$ is sufficiently smooth so that its derivatives are defined, then we can form new functions by taking derivatives f', f'', f''', $f^{(iv)}$.

4. One-to-one Correspondences

One-to-one and Onto. The only requirement on a function is that each argument can have one and only one value. However, it is possible for the function to take the same value for different arguments, cf. constant functions for example. Consider the functions from $\mathbb{R} \to \mathbb{R}$ given by $f_1(x) = x^2$, $f_2(x) = e^x$, $f_3(x) = x^3$, and $f_4(x) = x(x-1)(x+1)$. The first, f_1, maps the real line onto the non-negative numbers: for every $y \geq 0$, there is an $x \in \mathbb{R}$ such that $f_1(x) = y$. Moreover, for every $x \in \mathbb{R}$, f_1 takes the same value on $\pm x$. The function f_2 also maps the real line onto the non-negative numbers, but it does it "without repeats". If $f_2(x) = f_2(z)$, it must be true that $x = z$. Or put another way, if $x \neq z$, then $f_2(x) \neq f_2(z)$. Such a function is said to be "one-to-one" because every value is mapped from only one point. The function f_3 is also one-to-one, but in addition, it maps $\mathbb{R}$ onto $\mathbb{R}$ itself: for every $y \in \mathbb{R}$, there is an $x \in \mathbb{R}$ with $y = f_3(x)$. In contrast, f_4 maps $\mathbb{R}$ onto $\mathbb{R}$ but it is not on one-to-one as a quick sketch of its graph will show.

These different properties of a function are defined as follows. We say that a function $f: \mathcal{X} \to \mathcal{Y}$ is

- *onto* if the range and codomain are equal, that is $f(\mathcal{X}) = \mathcal{Y}$.
- *one-to-one* if $x_1 \neq x_2$ implies that $f(x_1) \neq f(x_2)$.

- *A one-to-one correspondence* if f is both onto and one-to-one.

As we saw above, two sets are said to have the same cardinality if there exists a one-to-one correspondence between them. If f is a one-to-one, then it is a one-to-one correspondence between its domain, $\mathcal{X}$, and its range, $f(\mathcal{X})$.

Inverse Functions. If f is a one-to-one correspondence between $\mathcal{X}$ and $\mathcal{Y}$, then, under f, each element of $\mathcal{X}$ corresponds to a unique element of $\mathcal{Y}$. Hence, we can define a function $f^{-1}\colon \mathcal{Y} \to \mathcal{X}$ that maps the other way. This is called the *inverse* of f, and if f has an inverse,[13] we say that f is *invertible*. In particular, we have that

- $f^{-1}(f(x)) = x$
- $f(f^{-1}(y)) = y$
- If $\mathcal{B} \subset \mathcal{Y}$, then the direct image of $\mathcal{B}$ under f^{-1}, $f^{-1}(\mathcal{B})$, is exactly the same as the inverse image of $\mathcal{B}$ under f defined above, also denoted by $f^{-1}(\mathcal{B})$.

13 Not every function has an inverse, only those that are one-to-one correspondences between their domain and codomain.

Despite all the notation, the idea is simple. If f maps $\mathcal{X}$ and $\mathcal{Y}$ by one-to-one correspondence, we think of each point in $\mathcal{X}$ connected by an arrow to one and only one point in $\mathcal{Y}$. The inverse function f^{-1} simply reverses the direction of the arrow.

Thought Question Suppose f has an inverse function and also has at least one derivative. Use the chain rule to find the derivative of f^{-1}.

Monotonic Functions. Real-valued functions defined on some subset $\mathcal{A}$ of $\mathbb{R}$ that do not decrease as their arguments increase are called increasing. More precisely, if $f\colon \mathcal{A} \to \mathbb{R}$ and if $x > y$ implies that $f(x) \geq f(y)$, we say that f is *increasing*. If $x > y$ implies $f(x) > f(y)$, we say that f is *strictly increasing*. The function $-1_{]-\infty,0[} + 1_{[0,1]} + 2 \cdot 1_{]1,\infty[}$ is increasing but not strictly increasing; the functions e^x, $\tan x$, $x/1{+}x$ are strictly increasing. Real-valued functions defined on some subset $\mathcal{A}$ of $\mathbb{R}$ that do not increase as their arguments increase are called decreasing More precisely, if $f\colon \mathcal{A} \to \mathbb{R}$ and if $x > y$ implies that $f(x) \leq f(y)$, we say that f is *decreasing*. If $x > y$ implies $f(x) < f(y)$, we say that f is *strictly decreasing*. The function $2 \cdot 1_{]-\infty,0[} + 1_{[0,1]} - 1_{]1,\infty[}$ is decreasing while e^{-x}, $1/1 + x$, and the function that maps $x \in \,]0,\infty[$ to $1/x$ are strictly decreasing. A function that is either increasing or decreasing is called *monotonic*. The functions sin, x^2, and e^{-x^2} are not monotonic.

Thought Question If $f\colon \mathbb{R} \to \mathbb{R}$ is a strictly increasing function with at least one derivative, is it necessarily a one-to-one correspondence between its domain and range? Why or why not?