

Applying Non-parametric Robust Bayesian Analysis to Non-Opinionated Judicial Neutrality

by

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Abstract

This paper explores the usefulness of robust Bayesian analysis in the context of an applied problem, finding priors to model judicial neutrality in an age discrimination case. We seek large classes of prior distributions without trivial bounds on the posterior probability of a key set, that is, without bounds that are independent of the data. Such an exploration shows qualitatively where the prior elicitation matters most, and quantitatively how sensitive the conclusions are to specified prior changes. The novel non-parametric classes proposed and studied here represent judicial neutrality and are reasonably wide, so that when a clear conclusion emerges from the data at hand, this is arguably *very reliable*.

Key Words: discrimination, elicitation, law, linearization, moment problem.

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1 Introduction

Robust Bayesian analysis (RBA), as championed by Berger and others (Berger, 1994) examines the maximum and minimum over a set of prior distributions, of a quantity of interest, like the posterior expectation of a function in the parameter space, for instance, the posterior probability of some set C . This paper explores what can be learned from such an analysis in the context of a specific example, the modeling of judicial neutrality in employment discrimination lawsuits.

What might be learned from a robust Bayesian analysis? We think there are at least two senses in which it can shed light on a standard Bayesian analysis that used a single, or a very few, prior distributions. The first is qualitative. If the class of prior distributions is unrestricted, for example if it contains both priors putting probability zero and probability one on the set C , the posterior probability will also be bounded by zero and one, independent of the data. This phenomenon, bounds attained whatever be the data, is referred in this paper as a set of priors leading to trivial bounds. At the other extreme, a class of prior distributions consisting of a single prior distribution is indistinguishable from an ordinary Bayesian analysis. However, if one can find a large class of prior distributions leading to non-trivial bounds, such an analysis can be informative about what aspects of the prior are particularly important to the determination of C 's posterior probability. This qualitative information, in turn, can help direct attention in elicitation to the aspects of the prior that matter most for the application.

The second kind of information that might be gleaned from an RBA is quantitative, namely how much the probability of C changes as the class expands in a qualitatively sensitive direction. This in turn can lead to a judgment of whether a Bayesian analysis is sufficiently robust to changes in details of the prior specification to be relied upon in a specific application with specific data. Robust Bayesian analyses, with reasonable wide classes of priors ("reasonable" meaning that dogmatic priors are excluded), are particularly well suited to judicial weighting of evidence.

While there has been a rich theoretical development in RBA (which this paper uses to guide the calculations), applications have been fewer. One reason is that in settings involving several parameters, the extrema over large classes of prior distributions tend to be difficult to find (analytically or computationally), and the bounds tend to be extreme. Kass and Greenhouse (1989) discuss the famous ECMO trial (Ware (1989)) from a robust Bayesian viewpoint. They report having calculated with 84 different priors, spanning what they claim is "a satis-

factory range of appropriate skeptical opinions,” and report reasonable robustness in stopping the trial.

Greenhouse and Wasserman (1995) uses an ϵ -contamination model in which the class of priors considered is

$$\{(1 - \epsilon)\pi_o + \epsilon q : q \in \mathcal{Q}\}$$

where π_o is the tentatively believed prior, and \mathcal{Q} is all possible prior distributions. As $\epsilon \downarrow 0$, the posterior probability of a set C approaches that under π_o ; as $\epsilon \uparrow 1$, the bounds on a set C become the trivial bounds 0 and 1. They find that a cancer trial was appropriately stopped, but that the ECMO decision is more equivocal. Finally, Greenhouse and Wasserman (1996) considers a testing framework in which they want to discern which of four models is favored by the data. They choose a main prior, and consider four variants of it to show robustness. A general and more radical approach that focus on upper and lower probabilities, is presented in Walley (1991).

This paper re-analyses the age-discrimination data of Kadane (1990). In that paper, four firing waves were studied using two-by-two doubly constrained contingency tables. Since the likelihood in this case has a single parameter with a natural interpretation, it is reasonable to hope that a broad class of priors might lead to reasonable bounds. This application has the difficulty, however, that the decision maker (judge or juror) is unavailable for prior elicitation. For this reason, it might be argued that RBA is unavoidable here. Hence we seek the prior of an idealized “judicially neutral” person.

Section 2 reviews the application and previous efforts at a robust analysis of it, which are still too narrow. Section 3 proposes some new classes of priors, and examines the conclusions that might be drawn from them. Section 4 concludes the paper with a discussion of the implications of our findings for the application, and for robust Bayesian analysis in general. The computational methods we used are described in the appendix.

2 Age Discrimination

The data come from a single company, which reduced its workforce in four firing waves, of 6/30/82, 11/30/82, 5/31/83 and 6/28/84, respectively. An employee was fired in the 4th wave, and sued the company claiming that persons whose age is over 40 were fired at uncommonly high rates. Table 1 gives the four 2×2

contingency tables. For example, in the first firing wave, 18 employees over 40 were fired, and none under forty were fired; among those not fired, 129 were over forty and 102 were under 40.

Kadane (1990) analyzed the data in Table 1 using 2×2 contingency tables. He took both the marginal distribution by age (over and under 40) and the marginal distribution by employment status (fired or retained) as fixed, since they are legally irrelevant. Thus both the age structure of the work force and the number fired are neutral happenstances, and the examination is based on who (by age) was chosen to be fired or retained.

The likelihood for a single firing wave is

$$f(n_{11} \mid \theta) = \frac{\binom{n}{n_{11}, n_{1+} - n_{11}, n_{+1} - n_{11}, n - n_{1+} - n_{+1} + n_{11}} e^{n_{11}\theta}}{\sum_{j=v}^w \binom{n}{j, n_{1+} - j, n_{+1} - j, n - n_{1+} - n_{+1} + j} e^{j\theta}}, \quad (1)$$

where $v = \max(0, n_{1+} + n_{+1} - n)$, $w = \min(n_{1+}, n_{+1})$. The parameter θ has the interpretation of being the log-odds-ratio, i.e. $\theta = \log(p_{11}p_{22}/p_{12}p_{21})$, the log-odds of being fired if an employee is over forty compared to one under forty. Therefore $\theta = 0$ means that the firing policy is age-neutral, $\theta > 0$ means that the firing policy disadvantages people older than 40, and conversely, $\theta < 0$ means that younger workers are disadvantaged. Since $\theta \leq 0$ is legal, while $\theta > 0$ is not, there are only two decisions available to the decision maker. Thus we concentrate on $P[\theta > 0]$. Notice, in passing, that one of the advantages of the Bayesian approach, preserved under the RBA, is its probability coherence. Thus the conclusions are invariant under smooth transformations of the parameter θ , although this is a very natural parameterization of the likelihood.

A hierarchical model might be useful to address the four firing waves jointly. Since there are only four observations at the second level of this hierarchy, inferences at the second level are quite sensitive to the assumptions made about the second level. Instead we address this issue by examining two extreme cases of such a hierarchical model: (i) treating firing waves as independent, leading to four separate analyses and (ii) treating the log-odds parameter θ as identical over all four firing waves, leading to a joint treatment of the four tables together. Varying amounts of assumed smoothing would lead to pulling the results of (i) toward the joint treatment results of (ii).

To be useful in court, a prior for this problem needs to represent not the real prior opinion of an expert witness, but rather an opinion that is judicially neutral, i.e. that does not bias the analysis in either direction. In some circumstances, one

might imagine eliciting the prior of a decision-maker. The institutional context of a legal case makes this impossible. However, some features of a suitable neutral prior are apparent. Kadane proposes symmetry for θ around zero and unimodality as reasonable features for such a neutral prior to have. Judicial neutrality might be compared to the “skeptical” prior proposed by Spiegelhalter, Freedman and Parmar (1994), intended “to formalize the belief that large treatment differences are unlikely” (p. 364). Thus the motivation of such a prior is to avoid being overenthusiastic about either treatment after the clinical trial is over, which is a different issue than judicial neutrality, which seeks to avoid prejudice against the position of either party. Carlin and Perez (2000) discuss the usefulness of classes of priors in biostatistical settings.

For convenience, Kadane chose the normal family with zero mean, and calculated posteriors using standard deviations of 1, 2, 4, 8, and ∞ . This is an example of parametric robust inference, in that the families of priors permitted are indexed by a parameter, here the standard deviation. Kadane put greatest stress on large standard deviations, infinity for waves II, III, and IV, and eight for wave I (in wave I, the zero in the table leads to an improper posterior under an unbounded uniform prior).

As explained above, this paper revisits the question of judicial neutrality, to see whether the techniques of RBA were useful in this context.

Perez and Pericchi (1994) noticed that the likelihood (1) belongs to the Exponential Family which can be written, in obvious notation as:

$$f(n_{11} \mid \theta) \propto \exp[n_{11}\theta - M(\theta)]g(\mathbf{n}),$$

so that a natural conjugate prior exists of the form,

$$\pi(\theta \mid n_0, x_0) \propto \exp[n_0 x_0 \theta - n_0 M(\theta)].$$

Perez and Pericchi (1994) explore this conjugate family, which has hyperparameters x_0 and n_0 . They suggest using $n_0 = 1$ and $x_0 = n_{1+}n_{+1}/n$, denoted C here. This choice is nearly, but not exactly, symmetric around zero. Consequently they suggest the even part of the conjugate prior, which is symmetric by construction (C_E). Additionally, they report the consequences of the Jeffreys prior, and its even part (denoted here J and J_E respectively) shown in Perez (1994) to be proper. These results of Kadane and Perez and Pericchi are summarized in Table 2, which shows a broad, general consistency in the results.

None of these reflect a class of priors, so, even together, they do not address fully the issue of the robustness of the inference. To address this, Perez and Pericchi study two kinds of classes of prior.

The first, following Edwards, Lindman and Savage (1963), and Dickey (1976), considers the class of all priors $\pi(\theta)$ satisfying

$$\frac{\pi(\theta)}{\xi(\theta)} \leq \lambda \quad \forall \theta, \quad (2)$$

where $\xi(\theta)$ is a fixed operational prior and $\lambda \geq 1$. Using as operational prior C, J, and N(0,1), they compute lower bounds for $P(\theta > 0)$ for several λ 's.

The second class studied by Perez and Pericchi uses an idea introduced by DeRobertis and Hartigan (1981). They consider the class of unnormalized prior distributions $\pi(\theta)$ satisfying $\ell(\theta) \leq \pi(\theta) \leq u(\theta)$ for almost all θ , where $\ell(\theta)$ and $u(\theta)$ are specified and need not integrate to unity. Using the choices $\ell(\theta) = N(0, \sigma^2)$, where σ^2 is chosen so that $\ell(0) = C(0)$ and $u(\theta) = \max_{\theta} (C_E(\theta)) * \frac{\ell(\theta)}{C_E(\theta)}$, they again derive lower bounds for the posterior probability that θ is positive. These results are given in Table 3. Again, they indicate a certain qualitative robustness. But these classes do not articulate well with the idea that the class of priors should represent judicial neutrality, since they do not require symmetry and unimodality around 0.

3 Robust Neutrality

The most natural class, from the viewpoint of judicial neutrality, is the class of unimodal priors symmetric around 0. However, as the analysis below shows, this class turns out to be too broad.

According to the Khinchine representation theorem (see Dharmadhikari and Joag-Dev. (1988), p. 10), every prior in this class can be represented as an arbitrary mixture of uniform densities symmetric around zero. Kadane (1990) remarks that a prior that puts all its weight uniformly on $[-\eta, \eta]$ will put posterior probability 1/2 on the set $\theta > 0$, (as $\eta \rightarrow 0$) regardless of what the data are. Denote this prior (really a limit of priors) as $U(0^+)$. Another important prior is $U(\infty-)$, found by taking a uniform prior on $[-m, m]$ as $m \rightarrow \infty$. Consider now the prior mixture of two uniforms:

$$\pi_p(\theta) = \begin{cases} p & U(0+) \\ 1-p & U(\infty-) \end{cases} \quad (3)$$

for some $p \in (0, 1]$. Also suppose there are no zeros in the data table (thus excluding Table I), which corresponds to $v < n_{11} < w$. Then as $|\theta| \rightarrow \infty$, it is easy to see that $f(n_{11} | \theta) \rightarrow 0$ exponentially fast. Now consider the posterior that results from the prior (3).

$$P\{\theta \leq 0 | n_{11}\} = \frac{(1/2)pf(n_{11} | \theta = 0)}{pf(n_{11} | \theta = 0) + (1-p) \lim_{m \rightarrow \infty} \int_{-m}^m \frac{1}{2m} f(n_{11} | \theta)} \quad (4)$$

But since $f(n_{11} | \theta) \rightarrow 0$ exponentially fast, the limit in the denominator of (4) is zero, so

$$P\{\theta \leq 0 | n_{11}\} = 1/2 \quad (5)$$

for all members of the class (3), irrespective of the data n_{11} . We were surprised by this property of the class (3).

This fact already suggests at least one qualitative result: a subclass of all unimodal priors symmetric around zero can lead to a non-trivial bound only if it avoids putting too much probability close to zero, and avoids allowing too much probability to be put on extremely high and low values of θ . This explains the failure of a number of our early attempts to restrict the class of unimodal, symmetric priors: (i) by fixing the height of π at 0,¹(ii) by bounding the variance of π from below, and (iii) by fixing a quantile of π . In each case, a member of (3) can be found to satisfy the constraint, thus showing that the class, even restricted, is uninterestingly broad, in the sense that 1/2 would be one of the possible values of $P\{\theta > 0\}$, regardless of the data. This demonstration that class (3) leads to trivial bounds cannot be taken as evidence that only class (3) leads to trivial bounds.

The priors $U(0+)$ and $U(\infty-)$ are not satisfactory representations of judicial neutrality. The former says essentially that the neutral arbitrator is sure, before hearing any evidence, that if there were age discrimination at all, its magnitude is

¹To see this, suppose that $\pi(0) \leq h$ for some fixed h . Consider a prior that puts $\pi(x) = h$ for $-\eta \leq x \leq \eta$, and puts $1 - 2h\eta$ probability on $U(\infty-)$. For each $\eta > 0$, such a prior has $P\{\theta \leq 0 | n_{11}\}$ nearly 1/2 independent of the data n_{11} . Hence as η decreases, this continues to be the case.

so small as to have a negligible effect. The prior $U(\infty-)$ is also unreasonable, as it puts all its predictive weight on v and w . Thus a neutral arbitrator holding this prior is sure, before seeing the data, that all the firings will be of employees under forty or all will be of employees over forty. Not only is this not generally the case, but it is not a good model for a neutral arbitrator either.

A referee proposed a class that restricts priors to be (i) symmetric around 0, (ii) unimodal, (iii) has bounded height at 0 and (iv) has a specified quantile. We agree that such a class can have non-trivial bounds. However we think that the joint elicitation required would be difficult. Also the interpretation of such a class would be less natural than the proposals below, as it depends on two parameters instead of just one.

Predictive distributions are a particularly useful way to think about the consequences of a prior distribution because it refers to what a neutral arbitrator might expect about the number of employees over forty fired, after learning the age structure of the work force and the number of employees to be fired. Berger (1994) interestingly points out that the predictive distribution is in fact the likelihood of the prior (for a fixed likelihood), and a limitation of some RBA studies is that robustness might be missing due to priors which have a very low (posterior) likelihood. In other words lack of robustness might be caused by priors which are ruled out by the data. Berger's insight seems to be consistent with the following principle in our situation: Neutrality might be considered in terms of not being too surprised at any way the data might come out. More formally, suppose that the prior is $\pi(\theta)$, and the likelihood is $f_i(\theta)$ where $i = n_{11}$ is the datum. Let

$$g_\pi(i) = \int_{-\infty}^{\infty} f_i(\theta) \pi(\theta) d\theta. \quad (6)$$

Then the neutral class A can be defined as

$$\begin{aligned} A = \{ \pi(\theta) : \pi(\theta) \text{ is unimodal and symmetric around 0,} \\ \text{and } g_\pi(i) \geq \epsilon \text{ for all } i, v \leq i \leq w \} \end{aligned} \quad (7)$$

The parameter ϵ of this class is then the minimum prior predictive probability of the possible data. The idea of this class is that it constrains the neutral arbitrator to have at least probability $\epsilon > 0$ on each possible data point. In other words, only priors which have a non-negligible likelihood, for all possible data, are allowed in our *neutrality class*. Every $\epsilon > 0$ prevents $U(\infty-)$ as a possible prior. To prevent $U(0+)$ from being a possible prior, it would be necessary to have

$$\epsilon > \min_i f(0 | i).$$

On the other hand, it is necessary to have $\epsilon < 1/(w - v + 1)$ in order for the class to be non-empty.

For each wave and for the combined waves as given in Table 1, there is a maximum value of epsilon, above which the class A is empty. For waves I to IV, and the combined waves, these maxima are .032, .016, .023, .041, and $8.57 \cdot 10^{-32}$, respectively. Thus any useful choice for epsilon must not exceed these numbers for the associated data set. It seems reasonable to allow epsilon to be one tenth or one hundredth of this maximum value, and to see the extent to which the posterior probability of positive θ varies as a result.

Another base that might be used is the height that the predictive distribution would have if it were uniform, namely $1/(w - v + 1)$ and then taking 1/10 or 1/100 of this base. For waves I to IV, and the combined waves, these heights are .053, .027, .036, .063, and 3.2×10^{-6} , respectively. For the individual waves, these numbers are larger than but the same order of magnitude as the maxima reported above. However, this idea is infeasible for the combined table, as it leads to an empty class.

Table 4 records the results for the data in Table 1. (The appendix to this paper shows how the computations were done). The computations in Table 4 show that the upper and lower bounds come together as the class of priors narrows by increasing epsilon, as expected. It also shows that each computation is unaffected by epsilons much smaller than one or two orders of magnitude below the maximum or the height the predictive distribution would have if it were uniform. Thus the calculations look quite insensitive to selection in class A with ϵ 's chosen as suggested above.

It may be, however, that predictive probabilities themselves are hard to think about, because they depend so much on the margins taken to be fixed. This consideration led to the construction of a second class of prior distributions, class B. The idea here is to constrain the ratio of predictive probabilities, i.e. so that

$$\frac{\max_i g_\pi(i)}{\min_i g_\pi(i)} \leq s \tag{8}$$

for some $s > 1$ (among π 's unimodal and symmetric around zero). Here there is a minimum value of s below which the class again goes empty. Respectively, those

values are 2.94, 4.52, 3.40, 3.57 and 77.1. Here again it makes sense to allow variations of one or two orders of magnitude (factors of 10 to 100). Again, the appendix shows how the calculations were done. Table 5 records the results.

Again, variations of one or two orders of magnitude on the minimum s do not affect the results. Thus these calculations confirm the results of Kadane (1990) and Perez and Pericchi (1994) that the calculations are robust. The classes considered are indeed very wide, allowing a plethora of different behavior, but at the same time obeying the natural requirement of being neutral and non-dogmatic, in order to be fair in the final judgement, whatever the outcome of the data.

A referee raises the question of whether the classes of priors specified here is data-dependent. The likelihood used in this paper is doubly-constrained; both the row and column sums are fixed. Hence, under one interpretation, the row and column sums are not data, and the classes of priors proposed here are not data-dependent. Another interpretation treats the row and column sums as data. In this sense the classes of priors proposed here are data-dependent.

4 Conclusions

The Robust Bayesian analyses performed here lead to two kinds of conclusions. Qualitatively, they show that there are two sensitive areas of prior elicitation, close to $\theta = 0$, and θ very large in absolute value. Neither of these comports well with the idea of judicial neutrality: not favoring either litigant and being open to being influenced by the data. Quantitatively, the analyses confirm the impression gained from the earlier studies, that for this particular application and data sets, the results are satisfactorily robust.

What have we learned from RBA to judicial weighting of the evidence? That a RBA is the natural implementation of the Bayesian approach to convey that a conclusion is very reliable.

On the other hand, what have we learned from this representative case study to RBA in general? That a RBA should take into account the likelihood of the priors in the class. This is encapsulated in the principle that classes that claim to model judicial neutrality, should yield non-negligible likelihoods of *any* possible outcome. It is our hope that this study will motivate similar principles in other realms of knowledge, and make closer to current judicial practice sensible Robust Bayesian analyses.

Appendix Finding the Extrema for Classes A and B

The data may consist of a single table, or of several tables. In either case, the sample space is discrete. For a single table, it consists of integers i such that $v \leq i \leq w$. For several tables indexed by t , it consists of a vector of integers of length T , $\mathbf{i} = (i_1, \dots, i_T)$ such that $v_t \leq i_t \leq w_t$ for all t , $1 \leq t \leq T$. Let N denote this sample space.

The likelihood function may then be written as $f_{\mathbf{i}}(\theta)$. The class A is then defined in equation (7), with the modification above in the case of more than one table. Using Khinchine's representation, the set of priors in A can be expressed as the set of distribution functions F satisfying

$$\pi(\theta) = \int_0^\infty \frac{1}{a} dF(a), \quad (9)$$

where F is a distribution function in the set $\mathcal{F} = \{F(\cdot) : \int_0^\infty dF(a) = \frac{1}{2}\}$,

$$\int_0^\infty f_{\mathbf{i}}^*(a) dF(a) \geq \epsilon > 0 \text{ for all } \mathbf{i} \in N \quad (10)$$

where

$$f_{\mathbf{i}}^*(a) = \frac{1}{a} \int_0^a (f_{\mathbf{i}}(-\theta) + f_{\mathbf{i}}(\theta)) d\theta, \quad a > 0$$

and $f_{\mathbf{i}}^*(0)$ is defined by continuity.

The quantity of interest is the probability of $\{\theta \geq 0\}$ posterior to observing \mathbf{i} when the prior is $\pi(\theta) \in A$. This can be rewritten as

$$P^\pi[\theta \geq 0 \mid \mathbf{i}] = \left[1 + \frac{\int_{\theta \geq 0} f_{\mathbf{i}}(-\theta) \pi(\theta) d\theta}{\int_{\theta \geq 0} f_{\mathbf{i}}(\theta) \pi(\theta) d\theta} \right]^{-1} \quad (11)$$

or, equivalently as

$$P^F[\theta \geq 0 \mid \mathbf{i}] = \left[1 + \frac{\int_0^\infty f_{\mathbf{i}}^1(a) dF(a)}{\int_0^\infty f_{\mathbf{i}}^2(a) dF(a)} \right]^{-1} \quad (12)$$

where

$$\begin{aligned} f_{\mathbf{i}}^1(a) &= \frac{1}{a} \int_0^a f_{\mathbf{i}}(-\theta) d\theta, \\ f_{\mathbf{i}}^2(a) &= \frac{1}{a} \int_0^a f_{\mathbf{i}}(\theta) d\theta, \\ \text{and} \quad &F(\cdot) \in \mathcal{F}. \end{aligned}$$

Then the supremum of the posterior probability that $\theta \geq 0$ can be written as

$$\begin{aligned} \sup_{\pi \in A} p^\pi(\theta \geq 0 \mid \mathbf{i}) &= \sup_{F \in \mathcal{F}} p^F(\theta \geq 0 \mid \mathbf{i}) \\ &= \left[1 + \inf_{F \in \mathcal{F}} \frac{\int_0^\infty f_{\mathbf{i}}^1(a) dF(a)}{\int_0^\infty f_{\mathbf{i}}^2(a) dF(a)} \right]^{-1}. \end{aligned} \quad (13)$$

By the linearization algorithm (Lavine, Wasserman and Wolpert, 1993), the infimum in (13) is the unique solution in λ of the equation

$$\inf_{F \in \mathcal{F}} \int_0^\infty [f_{\mathbf{i}}^1(a) - \lambda f_{\mathbf{i}}^2(a)] dF(a) = 0. \quad (14)$$

Once λ_0 has been found,

$$\sup_{\pi \in A} P^\pi(\theta \geq 0 \mid \mathbf{i}) = (1 + \lambda_0)^{-1}. \quad (15)$$

Using Kemperman (1987) (see also Salinetti (1994) and Liseo, Moreno and Salinetti (1994)), (14) can be rewritten

$$0 = \sup_{\substack{d_{\mathbf{i}} \geq 0 \\ \mathbf{i} \in N}} \left\{ \epsilon \sum_{\mathbf{i}' \in N} d_{\mathbf{i}'} + \inf_{F \in \mathcal{F}_0} \int_0^\infty \left[f_{\mathbf{i}}^1(a) - \lambda f_{\mathbf{i}}^2(a) - \sum_{\mathbf{i}' \in N} d_{\mathbf{i}'} f_{\mathbf{i}'}^*(a) \right] dF(a) \right\} \quad (16)$$

where \mathcal{F}_0 is the class $\{F(\cdot) : \int_0^\infty dF(a) = 1/2\}$.

While (16) may look formidable, and hence not a simplification, it has important consequences. First, the internal infimum occurs at an F that puts all its probability at a single point a . This means that the extremum will occur at a single uniform distribution for θ . Thus

$$\begin{aligned} &\inf_{F \in \mathcal{F}_0} \int_0^\infty \left[f_{\mathbf{i}}^1(a) - \lambda f_{\mathbf{i}}^2(a) - \sum_{\mathbf{i}' \in N} d_{\mathbf{i}'} f_{\mathbf{i}'}^*(a) \right] dF(a) \\ &= \inf_{a \geq 0} \frac{1}{2} \left[f_{\mathbf{i}}^1(a) - \lambda f_{\mathbf{i}}^2(a) - \sum_{\mathbf{i}' \in N} d_{\mathbf{i}'} f_{\mathbf{i}'}^*(a) \right], \end{aligned} \quad (17)$$

which permits reduction of (16) to

$$0 = \sup_{\substack{d_i \leq 0 \\ \mathbf{i} \in N}} \inf_{a \geq 0} \left\{ f_{\mathbf{i}}^1(a) - \lambda f_{\mathbf{i}}^2(a) - \sum_{\mathbf{i}' \in N} (f_{\mathbf{i}'}^*(a) - 2\epsilon) d_{\mathbf{i}'} \right\} = 0. \quad (18)$$

From (18), if for any i , $f_i^*(a) < 2\epsilon$, allowing the corresponding d_i to go to infinity results in a sup of infinity, so (18) cannot be satisfied. Hence the supremum is attained when $d_i = 0$ for all i . Thus (18) further simplifies to finding a value λ_0 of λ such that

$$\inf_{a \geq 0} \left\{ (f_{\mathbf{i}}^1(a) - \lambda f_{\mathbf{i}}^2(a)) I_{\{a \geq 0: f_{\mathbf{i}}^*(a) \geq 2\epsilon, \forall \mathbf{i} \in N\}}(a) \right\} = 0. \quad (19)$$

Now there are two cases to be considered separately. Since the supremum of $P^\pi(\theta \geq 0 \mid i)$ corresponds to a small value of λ , find the value of a for which $f_{\mathbf{i}}^1(a)/f_{\mathbf{i}}^2(a)$ is a minimum. If that value of a satisfies the constraint $\min_{\mathbf{i}} f_{\mathbf{i}}^*(a) > 2\epsilon$, then the supremum has been found. If not, then the constraint is binding. In this case, because $f_{\mathbf{i}}^1(a) - \lambda f_{\mathbf{i}}^2(a)$ is continuous in a , the infimum in (19) occurs when $f_{\mathbf{i}}^*(a) = 2\epsilon$ for some $\mathbf{i} \in N$.

Thus the search for a solution of (19) in the second case can be found at the points a at which

$$\min_{\mathbf{i} \in N} f_{\mathbf{i}}^*(a) = 2\epsilon, \quad (20)$$

and then

$$\lambda(a) = f_i^1(a)/f_i^2(a). \quad (21)$$

If there are several points a satisfying (20), the smallest $\lambda(a)$ in the set corresponds to the infimum in (13). This can be accomplished by a one-dimensional search over possible values a .

To find $\inf_{\pi \in A} p^\pi(\theta \geq 0 \mid i)$, simply reverse the roles of inf and sup in (13). This can be done by reversing the roles of $f_i^1(a)$ and $f_i^{(2)}(a)$ in each of the subsequent formulas.

Finding the extrema for the class B is quite similar to finding them for class A. Here the constraints (10) are replaced by the constraints (8), which are equivalent to

$$sg_\pi(\mathbf{i}') \geq g_\pi(\mathbf{i}'') \text{ for all } \mathbf{i}', \mathbf{i}'' \in N. \quad (22)$$

Hence equation (10) is replaced by

$$\int_0^\infty f_{\mathbf{i}', \mathbf{i}''}^*(a) dF(a) \geq 0 \text{ for all } \mathbf{i}', \mathbf{i}'' \in N, \text{ where} \quad (23)$$

$$f_{\mathbf{i}', \mathbf{i}''}^*(a) = \frac{1}{a} \int_0^a s(f_{\mathbf{i}'}(\theta) + f_{\mathbf{i}'}(-\theta)) - (f_{\mathbf{i}''}(\theta) + f_{\mathbf{i}''}(-\theta)) d\theta,$$

and $f_{\mathbf{i}', \mathbf{i}''}^*(0)$ is defined by continuity.

The same linearization can be applied, leading to the following analog of (16):

$$0 = \sup_{\substack{d_{\mathbf{i}', \mathbf{i}''} \geq 0 \\ \mathbf{i}', \mathbf{i}'' \in N}} \inf_{F \in \mathcal{F}_0} \int_0^\infty \left[f_{\mathbf{i}', \mathbf{i}''}^1(a) - \lambda f_{\mathbf{i}', \mathbf{i}''}^2(a) - \sum_{\mathbf{i}', \mathbf{i}'' \in N} d_{\mathbf{i}', \mathbf{i}''} f_{\mathbf{i}', \mathbf{i}''}^*(a) \right] dF(a) \quad (24)$$

By exactly the same arguments, this results, in the case in which the constraints are binding, in finding the set of a 's such that

$$\min_{\mathbf{i}', \mathbf{i}''} f_{\mathbf{i}', \mathbf{i}''}^*(a) = 0$$

and then choosing the smallest $\lambda_0(a)$ from the resulting set. Again, the infimum over the class B is found by reversing the roles of f^1 and f^2 .

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Table 1: Ages of those fired and retained in four firing waves: Notation and Data

		Cell			
Notation:		1,1	1,2	2,1	2,2
Numbers		n_{11}	n_{12}	n_{21}	n_{22}
Probabilities		p_{11}	p_{12}	p_{21}	p_{22}
Wave	I	18	0	129	102
	II	26	10	105	83
	III	13	14	92	66
	IV	13	2	81	52
Category		Fired, 40+	Fired, 40-	Retained, 40+	Retained, 40-

(from Kadane (1990))

Table 2: Posterior Probability of $\theta > 0$ (older than 40 disadvantaged) as a function of the prior:

Prior	Wave				Combined
	I	II	III	IV	
$N(0, 1)$	1.000	.960	.183	.965	.999
$N(0, 2^2)$	1.000	.967	.169	.981	.999
$N(0, 4^2)$	1.000	.969	.165	.985	.999
$N(0, 8^2)$	1.000	.970	.164	.986	.999
$N(0, \infty)$	NA	.970	.164	.987	.999
C	.997	.906	.248	.930	.984
C_E	.997	.904	.246	.925	.984
J	1.000	.968	.165	.980	.998
J_E	1.000	.969	.170	.984	.999

(from Kadane (1990) and Perez and Pericchi (1993))

Table 3: Lower bound for the Posterior Probability that $\theta > 0$ as a function of the class of prior

Operational		Wave				
Prior	λ	I	II	III	IV	Combined
J	1.5	1.000	.952	.116	.971	.997
	2	1.000	.937	.090	.961	.996
	3	1.000	.909	.062	.944	.993
	4	1.000	.882	.047	.926	.991
	5	1.000	.857	.038	.909	.989
C	1.5	1.000	.865	.128	.899	.976
	2	1.000	.828	.099	.896	.969
	3	1.000	.763	.069	.816	.953
	4	1.000	.707	.052	.769	.939
	5	1.000	.658	.042	.939	.925
N(0,1)	1.5	1.000	.942	.132	.949	.998
	2	1.000	.924	.103	.933	.997
	3	1.000	.891	.071	.903	.996
	4	1.000	.859	.054	.875	.995
	5	1.000	.830	.044	.848	.993
DeRobertis/						
Hartigan		.996	.893	.268	.916	.982
Class						

(From Perez and Pericchi (1993))

Table 4: Upper and lower bounds on the posterior probability of age discrimination for Class A as a function of ϵ .

<u>Epsilon</u>	<u>Inf.</u>	<u>Sup</u>	<u>Epsilon</u>	<u>Inf.</u>	<u>Sup</u>
$4 \cdot 10^{-8}$.50	1.00	10^{-16}	.50	.97
$5 \cdot 10^{-8}$.69		$5 \cdot 10^{-16}$.67	
10^{-7}	.86		10^{-15}	.72	
10^{-6}	.98		10^{-14}	.83	
10^{-5}	.997		10^{-12}	.92	
			10^{-8}	.97	
Wave I			Wave II		

<u>Epsilon</u>	<u>Inf.</u>	<u>Sup</u>	<u>Epsilon</u>	<u>Inf.</u>	<u>Sup</u>
10^{-11}	.17	.50	10^{-8}	.50	.99
$5 \cdot 10^{-11}$.45	10^{-7}	.64	
10^{-10}		.32	10^{-6}	.86	
10^{-8}		.20	10^{-5}	.94	
10^{-6}		.17	.011	.98	
Wave III			Wave IV		

<u>Epsilon</u>	<u>Inf.</u>	<u>Sup</u>
10^{-42}	.50	.999
10^{-41}	.68	
10^{-40}	.81	
10^{-39}	.89	
10^{-37}	.96	
10^{-35}	.98	
Combined Waves		

Table 5: Upper and lower bounds on the posterior probability of age discrimination for Class B as a function of s .

s	<u>Inf.</u>	<u>Sup</u>	s	<u>Inf.</u>	<u>Sup</u>
10^5	.99	1.00	10^7	.97	.97
10^6	.92		10^{10}	.95	
$3 \cdot 10^6$.77		10^{12}	.90	
$4.5 \cdot 10^6$.61		10^{14}	.74	
$5 \cdot 10^6$.5		10^{15}	.54	
Wave I			Wave II		

s	<u>Inf.</u>	<u>Sup</u>	s	<u>Inf.</u>	<u>Sup</u>
10^6	.17	.18	10^3	.97	.99
10^8		.24	10^5	.89	
10^9		.31	10^6	.75	
10^{10}		.41	$2 \cdot 10^6$.66	
$3 \cdot 10^{10}$.50	$3.2 \cdot 10^6$.54	
Wave III			Wave IV		

s	<u>Inf.</u>	<u>Sup</u>
10^{24}	.998	.999
10^{30}	.992	
10^{34}	.957	
10^{36}	.886	
10^{37}	.810	
10^{38}	.680	
$3 \cdot 10^{38}$.569	
$4 \cdot 10^{38}$.500	

Combined Waves