Carnegie Mellon University Department of Statistics & Data Science

36-708 Statistical Machine Learning Homework #4 Solutions

DUE: April 19, 2019

Problem 1 [5 pts.]

Consider the directed graph with vertices $V = \{X_1, X_2, X_3, X_4, X_5\}$ and edge set $E = \{(1,3),(2,3),(3,4),(3,5)\}$.

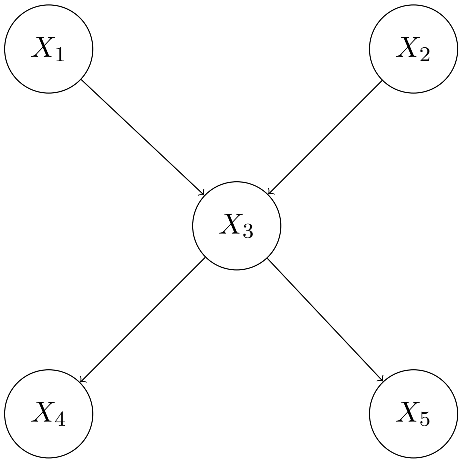
(a)**[2 pts.]** List all the independence statements implied by this graph.

(b)**[1 pts.]** Find the causal distribution $p(x_4|\text{set } x_3 = s)$.

(c)**[2 pts.]** Find the implied undirected graph for these random variables. Which independence statements get lost in the undirected graph (if any)?

Solution.

The graph can be visualized as follows:



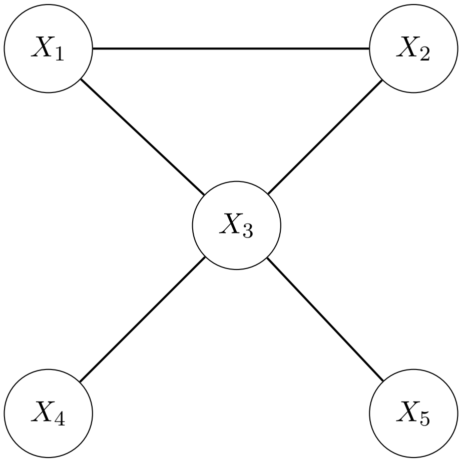
(a) The independence statements implied are the following:

- $X_1 \perp\!\!\!\perp X_2$;
- $X_4 \perp\!\!\!\perp \{X_1, X_2\}|X_3$ and $X_5 \perp\!\!\!\perp \{X_1, X_2\}|X_3$;
- $X_5 \perp\!\!\!\perp X_4|X_3$.

(b) Given that we set $X_3 = X_3$, then by the independence highlighted above X_1 and X_2 can be dropped from the graph. Hence we have that:

$$p(x_4|\text{set } x_3 = s) = \int p_*(x_4, x_5)dx_5 = \int p(x_4|x_3 = s)p(x_5|x_3 = s)dx_5 = p(x_4|x_3 = s)$$

(c) The moralized graph becomes:



We lose the (unconditional) independence between X_1 and X_2 during the moralization process, while all the others are retained.

Problem 2 [20 pts.]

Let $d \geq 2$, and let $X_1, \ldots, X_n \sim P$ where $X_i = (X_i(1), \ldots, X_i(d)) \in \mathbb{R}^d$. Assume that the coordinates of X_i are independent. Further, assume that $X_i(j) \sim \text{Bernoulli}(p_j)$ where $0 < c \leq p_j \leq C < 1$. Let $\mathcal{P}$ be all such distributions. Let

$$R_n = \inf_{\hat{p}} \sup_{P \in \mathcal{P}} \mathbb{E}_P \|\hat{p} - p\|_\infty.$$

Find lower and upper bounds on the minimax risk.

Solution.

For **upper bound**, consider an estimator $\overline{X}$ for estimating p. Then $\bar{X} - p$ is sub-Gaussian with parameter $\sigma^2 = \frac{1}{4n}$. Hence, from the lemma below,

$$\mathbb{E}\|\bar{X} - p\|_\infty = \mathbb{E}\left[\max_{1 \leq i \leq d} |\bar{X} - p|_i\right] \leq \frac{1}{2\sqrt{n}} \sqrt{2\log(2d)}.$$

And since $d \geq 2$, we have $2\log(2d) \leq 4 \log d$, so,

$$\inf_{\hat{p}} \sup_{P \in \mathcal{P}} \mathbb{E}_P \|\hat{p} - p\|_\infty \leq \mathbb{E}\|\bar{X} - p\|_\infty \leq \sqrt{\frac{\log d}{n}}.$$

For **lower bound**, let $\alpha = \sqrt{\frac{\log d}{16n}}$, $p^{(0)} = (c, \ldots, c) \in \mathbb{R}^d$.

And for $1 \leq j \leq d$, construct d-dimensional vector parameter

$$p_j := (p_j^{(i)}, i = 1, \cdots d) = (c, \cdots, c, \underbrace{c + \alpha}_{j\text{th}}, c, \cdots, c) \in \mathbb{R}^d,$$

then $\|p_j - p_k\|_\infty = \alpha$ for $j \neq k$, so

$$\min_{j \neq k} \|p_j - p_k\|_\infty = \alpha.$$

Let P_j be the multivariate Bernoulli with parameter p_j, then it's a product of uni-variate Bernoulli's, $P_j = \prod_{i=1}^d P_j^{(i)}$, where $P_j^{(i)}$ is uni-variate Bernoulli with parameter $p_j^{(i)}$. Then

$$KL(P^{(j)}, P^{(k)}) = \sum_{i=1}^d KL(P_i^{(j)}, P_i^{(k)}) = KL(P_j^{(j)}, P_j^{(k)}) + KL(P_k^{(j)}, P_k^{(k)}).$$

since they only differ in two terms: $i = j$ or $i = k$. Let uni-variate Beroulli with parameter c as P_c, then,

$$\begin{aligned} KL(P^{(j)}, P^{(k)}) &= KL(P_j^{(j)}, P_j^{(k)}) + KL(P_k^{(j)}, P_k^{(k)}) \\ &= \sum_x P_{c+\alpha} \log(\frac{P_{c+\alpha}}{P_c}) + \sum_x P_c \log(\frac{P_c}{P_{c+\alpha}}) \\ &= \sum_x (P_{c+\alpha} - P_c) \log(\frac{P_{c+\alpha}}{P_c}) \\ &= \alpha \log \frac{(\alpha + c)(1 - c)}{(1 - \alpha - c)c} \\ &\leq C\alpha^2, \quad \text{by Taylor expansion.} \end{aligned}$$

(Fano's method)

$$\begin{aligned}\max_{j\neq k} KL(P_j, P_k) &= \max_{j\neq k}\frac{1}{2}||\mu_j - \mu_k||_2^2 = C\alpha^2\\ &= \frac{\log d}{8n} \leq \frac{\log(d+1)}{4n}.\end{aligned}$$

Hence by Corollary 13 in the minimax notes where $N = d+1$,

$$\inf_{\hat{p}}\sup_{P\in\mathcal{P}} \mathbb{E}_P||\hat{p} - p||_\infty \geq \frac{\alpha}{4} = \sqrt{\frac{\log d}{256n}}.$$

Note: following the same construction of $p^{(i)}$ and KL distance bound, lower bounds for minimax with the same rate with respect to d and n can also be derived using Theorem 12 or Theorem 14.

Lemma 1 *(Maximal inequality for subgaussian random variables)*
Let $\{X_i\}_{1\leq i\leq n}$ *be sub-Gaussian variables with parameter* σ^2*, then*

$$\mathbb{E}\left[\max_{1\leq i\leq n} X_i\right] \leq \sigma\sqrt{2\log n} \text{ and } \mathbb{E}\left[\max_{1\leq i\leq n}|X_i|\right] \leq \sigma\sqrt{2\log(2n)}.$$

It's covered in Advanced stats.
See http://www.stat.cmu.edu/~arinaldo/Teaching/36755/F17/Scribed_Lectures/F17_0911.pdf.

Problem 3 [20 pts.]

Let $\{p_\theta : \theta \in \Theta\}$ where $\Theta \subset \mathbb{R}$ be a parametric model. Suppose that the model satisfies the usual regularity conditions. In particular, the Fisher information $I(\theta)$ is positive and smooth and the mle has the usual nice properties. Let the loss function be $L(\hat{\theta}, \theta) = H(p_{\hat{\theta}}, p_\theta)$ where H denotes Hellinger distance. Find the minimax rate.

Solution.

Assume that the densities in $\{p_\theta : \theta \in \Theta\}$ is differentiable in quadratic mean (QMD) as equation (27) in the Minimax note, which says the Hellinger distance between $p_{\theta+h}, p_\theta$ can be approximated by first order Taylor expansion on h. In other words, the loss with Hellinger distance is then approximately equivalent with that with squared loss on θ,

$$H^2(p_{\theta+h}, p_\theta) = \frac{1}{8}||h||^2 I(\theta) + o(||h||^2).$$

Proof see https://www.stat.berkeley.edu/ bartlett/courses/2013spring-stat210b/notes/25notes.pdf. So the minimax rate for Hellinger loss is the same as for squared risk $R(\hat{\theta}, \theta) = ||\hat{\theta} - \theta||^2$,

$$\inf_{\hat{\theta}} \sup_{\theta \in \Theta} H^2(p_{\hat{\theta}}, p_\theta) = O(\inf_{\hat{\theta}} \sup_{\theta \in \Theta} R(\hat{\theta}_n^{mle}, \theta)).$$

Next, we prove that the minimax rate for squared loss is achieved by MLE estimator. For a fixed true parameter θ, denote the MLE as $\hat{\theta}_n^{mle}$ from sample $X_1, \cdots, X_n \sim P_\theta$. By MLE asymptotic distribution,

$$\sqrt{n}(\hat{\theta}_n^{mle} - \theta) \to N(0, I^{-1}(\theta).$$

So the squared risk for the MLE is,

$$R(\hat{\theta}_n^{mle}, \theta) = Var(\hat{\theta}_n^{mle}) + bias^2 \to I^{-1}(\theta)/n.$$

For any other estimator $\hat{\theta}$, by theorem 17 with $\psi(x) = x$ and l as the square loss, under the QMD condition, the squared risk is lower bounded by that for MLE,

$$R(T_n, \theta) = Var(T_n) + bias^2 \geq Var(T_n) \geq Var(U) = I^{-1}(\theta)/n = R(\hat{\theta}_n^{mle}, \theta).$$

This lower bound holds for all $\theta \in \Theta$. So the minimax risk for squared loss is achieved by the MLE,

$$\inf_{\hat{\theta}} \sup_{\theta \in \Theta} R(\hat{\theta}, \theta) = \sup_{\theta \in \Theta} R(\hat{\theta}_n^{mle}, \theta) \to \frac{1}{n} \sup_{\theta \in \Theta} I^{-1}(\theta).$$

Therefore,

$$\inf_{\hat{\theta}} \sup_{\theta \in \Theta} H^2(p_{\hat{\theta}}, p_\theta) = O(\frac{1}{n}),$$

or equivalently,

$$\inf_{\hat{\theta}} \sup_{\theta \in \Theta} H(p_{\hat{\theta}}, p_\theta) = O(n^{-1/2}).$$

Alternative solution (thank Tim Barry for the ideas)
We first derive an upper bound for the KL distance between arbitrary $p_{\theta+h}, p_\theta$,

$$\begin{aligned}
KL(p_{\theta+h}, p_\theta) &= \int \log \frac{p_\theta}{p_{\theta+h}} p_\theta dx \\
&= \int \log p_\theta - \log p_{\theta+h} p_\theta dx \\
&= \int \log p_\theta - (\log p_\theta + h\frac{\partial \log p_\theta}{\partial\theta} + h^2 \frac{\partial^2 \log p_\theta}{\partial \theta^2} + o(h^2))p_\theta dx \\
&= h^2(-\int \frac{\partial^2 \log p_\theta}{\partial\theta^2} p_\theta dx) + o(h^2) \\
&\leq CI(\theta)h^2,
\end{aligned}$$

where we assume good properties for the density to allow swapping the derivative and integral, so that,

$$\int \frac{\partial \log p_\theta}{\partial\theta} p_\theta dx = \int \frac{1}{p_\theta}\frac{\partial p_\theta}{\partial\theta} p_\theta dx = \int \frac{\partial p_\theta}{\partial \theta} dx = \frac{\partial(\int p_\theta dx)}{\partial\theta} = 0.$$

For **upper bound**, consider MLE estimator,

$$H(p_{\hat\theta^{mle}}, p_\theta) \leq \sqrt{KL(p_{\hat\theta^{mle}}, p_\theta)} \leq \sqrt{C(\hat\theta^{mle} - \theta)^2 I(\theta)},$$

By MLE asymptotic distribution,

$$\hat\theta^{mle} - \theta \to N(0, I^{-1}(\theta)/n),$$

thus,

$$\inf_{\hat\theta}\sup_{\theta\in\Theta} H(p_{\hat\theta}, p_\theta) \leq H(p_{\hat\theta^{mle}}, p_\theta) \leq \sqrt{CI(\theta)(Var(\hat\theta_n^{mle}) + bias^2)} \to O(n^{-1/2}).$$

For **lower bound**, consider two distribution p_θ and $p_{\theta+h}$, where $h = \sqrt{\frac{\log 2}{CI(\theta)n}}$, then by QMD condition,

$$H(p_{\theta+h}, p_\theta) = Ch = O(n^{-1/2}).$$

And the KL distance,

$$KL(p_{\theta+h}, p_\theta) \leq CI(\theta)h^2 \leq \frac{\log 2}{n}.$$

Hence by Corollary 5 in the minimax notes,

$$\inf_{\hat\theta}\sup_{\theta\in\Theta} H(p_{\hat\theta}, p_\theta) = O(n^{-1/2}).$$

Problem 4 [15 pts.]

Let $Y = (Y_1, \ldots, Y_d) \sim N(\theta, I)$ where $\theta = (\theta_1, \ldots, \theta_d)$. Assume that $\theta \in \Theta = \{\theta \in \mathbb{R}^d : \|\theta\|_0 \leq 1\}$. Let

$$R_d = \inf_{\widehat{\theta}} \sup_{\theta \in \Theta} \mathbb{E}_\theta \|\widehat{\theta} - \theta\|^2.$$

Show that $c \log d \leq R_d \leq C \log d$ for some constants c and C.

Solution.

For the upper bound, we first prove a high-probability bound lemma for the maximum of Gaussians (from 36-705, Lecture 27, Fall 2018):

Lemma 2 *Suppose that, $\epsilon_1, \ldots, \epsilon_d \sim N(0, \sigma^2)$ then with probability at least $1 - \delta$,*

$$\max_{i=1}^{d} |\epsilon_i| \leq \sigma \sqrt{2 \log(2d/\delta)}.$$

Proof. By the Gaussian tail bound, if $\epsilon \sim N(0, \sigma^2)$:

$$\mathbb{P}(|\epsilon| \geq t) \leq 2 \exp(-t^2/(2\sigma^2)),$$

By using the union bound:

$$\mathbb{P}(\max_i |\epsilon_i| \geq t) \leq 2d \exp(-t^2/(2\sigma^2)),$$

By setting $2d \exp(-t^2/(2\sigma^2)) = \delta$ we obtain the lemma.

Now, we assume $\hat{\theta}$ to be the hard-thresholding estimator, defined as:

$$\widehat{\theta_i} = y_i \mathbb{I}(|y_i| \geq t), \quad \forall\ i \in \{1, \ldots, d\},$$

We have the following theorem (from 36-705, Lecture 27, Fall 2018):

Theorem 3 *Suppose we choose the threshold:*

$$t = 2\sigma \sqrt{2 \log(2d/\delta)},$$

then with probability at least $1 - \delta$,

$$\|\widehat{\theta} - \theta\|_2^2 \leq 9 \sum_{i=1}^{d} \min\left\{\theta_i^2, \frac{t^2}{4}\right\} \leq Ct^2$$

For some $C_1 > 0 \in \mathbb{R}$.

Proof. We condition on the event from the previous lemma, i.e. that

$$\max_{i=1}^{d} |\epsilon_i| \leq \sigma \sqrt{2 \log(2d/\delta)} \leq \frac{t}{2}.$$

Now, observe that,

$$\|\widehat{\theta} - \theta\|_2^2 = \sum_{i=1}^{d} (\widehat{\theta_i} - \theta_i)^2,$$

so we can consider each co-ordinate separately. Let us consider some cases:

1. If for any co-ordinate $|\theta_i| \le \frac{t}{2}$ our estimate is 0, so our risk for that coordinate is simply θ_i^2.
2. If $|\theta_i| \ge \frac{3t}{2}$ our estimate is simply $\widehat{\theta}_i = y_i$ so our risk is simply $\epsilon_i^2 \le \frac{t^2}{4}$.
3. If $\frac{t}{2} \le |\theta_i| \le \frac{3t}{2}$, then our risk,

$$(\widehat{\theta}_i - \theta_i)^2 = (y_i \mathbb{I}(|y_i| \ge t) - \theta_i)^2 = \theta_i^2 \mathbb{I}(|y_i| < t) + \epsilon_i^2 \mathbb{I}(|y_i| \ge t) \le \max\{\epsilon_i^2, \theta_i^2\} \le \frac{9t^2}{4}.$$

Putting these together we see that,

$$\|\widehat{\theta} - \theta\|_2^2 \le 9 \sum_{i=1}^d \min\left\{\theta_i^2, \frac{t^2}{4}\right\} = 9 \sum_{\theta_i = 0} \min\left\{\theta_i^2, \frac{t^2}{4}\right\} + 9 \sum_{\theta_i \ne 0} \min\left\{\theta_i^2, \frac{t^2}{4}\right\} \le C_1 t^2$$

We also have that, under the same assumptions of the theorems above:

$$\|\widehat{\theta} - \theta\|_2^2 = \sum_{i=1}^d (y_i \mathbb{I}(|y_i| \ge t) - \theta_i)^2 \le C_2 d \max_{i=1,..,d} (t - \theta_i)^2 \le C_2 d t^2$$

And so we have that:

$$\begin{aligned}
\mathbb{E}\left[\|\widehat{\theta} - \theta\|_2^2\right] &= \int_0^\infty \mathbb{P}\left(\|\widehat{\theta} - \theta\|_2^2 > x\right) dx \\
&= \int_0^{C_1 t^2} \mathbb{P}\left(\|\widehat{\theta} - \theta\|_2^2 > x\right) dx + \int_{C_1 t^2}^{C_2 d t^2} \mathbb{P}\left(\|\widehat{\theta} - \theta\|_2^2 > x\right) dx \\
&\le \int_0^{C_1 t^2} (1 - \delta) dx + \int_{C_1 t^2}^{C_2 d t^2} \delta dx \\
&\le C_1 t^2 (1 - \delta) + t^2 (C_2 d - 1) \delta \le C_3 t^2 \approx C_3 \log d
\end{aligned}$$

For the lower bound, let $P_j = N(\theta_j, I)$ where $\theta_0 = (0, \ldots, 0)$ and θ_j is the d-dimensional vector ($d \ge 8$) where

$$\theta_j(k) = \begin{cases} \sqrt{\frac{\log d}{32}} & k = j \\ 0 & k \ne j. \end{cases}$$

P_0 is absolutely continuous wrt each other distribution and for all $j = 1, \ldots, d$, we claim

$$KL(P_j, P_0) \le \frac{\log d}{16}.$$

The statement above actually works for any P_j, P_i with $i \ne j$. Let $P_i = N(\mu_i, I)$, then:

$$\begin{aligned}
KL(P_j,\, P_k) &= \int \frac{1}{2} \left(||x - \mu_k||_2^2 - ||x - \mu_j||_2^2\right) P_j(x) dx \\
&= \frac{1}{2} ||\mu_j - \mu_k||_2^2.
\end{aligned}$$

Which implies:

$$\max_{j \ne k} KL(P_j,\, P_k) = \max_{j \ne k} \frac{1}{2} ||\mu_j - \mu_k||_2^2 = 2 \left(\sqrt{\frac{\log d}{32}}\right)^2 = \frac{\log d}{16}$$

It then follows that:

$$\frac{1}{d}\sum_{j=1}^{d} KL(P_j, P_0) \leq \frac{\log d}{16}.$$

Therefore, by Tsybakov's bound,

$$\begin{aligned}
R_d &\geq \frac{s}{16} \\
&= \frac{\max_{j \neq k} \|\theta_j - \theta_k\|_2^2}{16} \\
&= \frac{2\sqrt{\frac{\log d}{32}}^2}{16} \\
&\geq c \log d.
\end{aligned}$$

Problem 5 [20 pts.]

Let $X_1, \dots, X_n \sim F$ where F is some distribution on $\mathbb{R}$. Suppose we put a Dirichlet process prior on F:

$$F \sim \text{DP}(\alpha, F_0).$$

(a) **(10 pts.)** Recall the stick-breaking construction. Show that $\mathbb{E}\big(\sum_{j=1}^{\infty} W_j \big) = 1$.

(b) **(10 pts.)** Simulate $n = 10$ data points from a $N(0,1)$. Try three values of α: namely, $\alpha = .1$, $\alpha = 1$ and $\alpha = 10$. Compute the 95 percent Bayesian confidence band and the 95 percent DKW band. Plot the results for one example. Now repeat the simulation 1,000 times and report the coverage probability for each confidence band.

Solution.

(a) We start by showing $\mathbb{P}(\sum_{j=1}^{\infty} W_j = 1) = 1$. The expectation will then follow easily. First we prove a series of lemmas.

Lemma 4 *For all $n \in \mathbb{N}$,*

$$1 - \sum_{j=1}^{n} W_j = \prod_{j=1}^{n}(1 - V_j).$$

Proof. (by induction)
Base case. $k = 1$

$$1 - W_1 = 1 - V_1. \checkmark$$

Inductive hypothesis. Now assume that

$$1 - \sum_{j=1}^{n-1} W_j = \prod_{j=1}^{n-1}(1 - V_j).$$

Inductive step.

$$\begin{aligned}
1 - \sum_{j=1}^{n} W_j &= 1 - \sum_{j=1}^{n-1} W_j - W_n \\
&= \prod_{j=1}^{n-1}(1 - V_j) - V_n \prod_{j=1}^{n-1}(1 - V_j) \\
&= (1 - V_n) \prod_{j=1}^{n-1}(1 - V_j) \\
&= \prod_{j=1}^{n}(1 - V_j). \checkmark \qquad (1)
\end{aligned}$$

Lemma 5 *Let $v_1, v_2, \dots$ be a sequence such that $0 < v_j < 1$ for all j. Then*

$$\prod_{j=1}^{\infty}(1 - v_j) > 0 \quad \textit{if and only if} \quad \sum_{j=1}^{\infty} v_j < \infty.$$

Proof. First notice that

$$-\log\prod_{j=1}^{\infty}(1-v_j) = -\sum_{j=1}^{\infty}\log(1-v_j),$$

and thus

$$\prod_{j=1}^{\infty}(1-v_j) > 0 \Longleftrightarrow -\sum_{j=1}^{\infty}\log(1-v_j) < \infty.$$

We now have

$$\Big\{-\log(1-v_j)\Big\}_{j\in\mathbb{N}} > 0 \quad\text{and}\quad \{v_j\}_{j\in\mathbb{N}} > 0,$$

so we can use the Limit Comparison test to prove that

$$-\sum_{j=1}^{\infty}\log(1-v_j) < \infty \Longleftrightarrow \sum_{j=1}^{\infty}v_j < \infty.$$

Since both series diverge when $v_j \not\longrightarrow 0$, it suffices to consider only the sequences where $v_j \longrightarrow 0$.

$$\lim_{v_j\to 0}\frac{-\log(1-v_j)}{v_j} \overset{L'H}{=} \lim_{v_j\to 0}\frac{1}{1-v_j} \tag{2}$$

$$= 1. \tag{3}$$

Hence,

$$-\sum_{j=1}^{\infty}\log(1-v_j) < \infty \quad\text{if and only if}\quad \sum_{j=1}^{\infty}v_j < \infty,$$

and thus,

$$\prod_{j=1}^{\infty}(1-v_j) > 0 \quad\text{if and only if}\quad \sum_{j=1}^{\infty}v_j < \infty.$$

Corollary 6 *Let $v_1, v_2, \dots$ be a sequence such that $0 < v_j < 1$ for all j. Then*

$$\prod_{j=1}^{\infty}(1-v_j) = 0 \quad\textit{if and only if}\quad \sum_{j=1}^{\infty}v_j = \infty.$$

Lemma 7 *(Borel-Cantelli) If $\sum_{j=1}^{\infty}\mathbb{P}(V_j > \epsilon) = \infty$ for some $\epsilon > 0$, then $\mathbb{P}(V_j > \epsilon \ \text{i.o.}) = 1$.*

Now since

$$V_j \sim \text{Beta}(1,\alpha), \quad j = 1, 2, \dots$$

we have

$$\mathbb{P}(V_j > \epsilon) > 0 \ \text{ for all } \epsilon \in (0,1) \ \text{ and } \ j = 1, 2, \dots \tag{4}$$

since the beta distribution puts positive mass over its entire support $(0,1)$, and now (4) implies

$$\sum_{j=1}^{\infty}\mathbb{P}(V_j > \epsilon) = \infty \ \text{ for all } \epsilon \in (0,1). \tag{5}$$

So altogether,

$$\begin{aligned}
\mathbb{P}\Big(\sum_{j=1}^{\infty} W_j = 1\Big) &\overset{\text{Lemma 2}}{=} \mathbb{P}\Big(\prod_{j=1}^{\infty}(1 - V_j) = 0\Big) \\
&\overset{\text{Cor. 4}}{=} \mathbb{P}\Big(\sum_{j=1}^{\infty} V_j = \infty\Big) \\
&\geq \mathbb{P}(V_j > \epsilon \ \ i.o.) \qquad (6) \\
&\overset{\text{Lemma 5}}{=} 1,
\end{aligned}$$

where the inequality comes from the fact

$$\Big\{\sum_{j=1}^{\infty} V_j = \infty\Big\} \supset \big\{V_j > \epsilon \ \ i.o.\big\}.$$

Thus,

$$\mathbb{P}\Big(\sum_{j=1}^{\infty} W_j = 1\Big) = 1. \ \blacksquare$$

It follows that

$$\begin{aligned}
\mathbb{E}\Big(\sum_{j=1}^{\infty} W_j\Big) &= \int_{\mathbb{R}} \sum_{j=1}^{\infty} W_j dF \\
&= \mathbb{P}\Big(\sum_{j=1}^{\infty} W_j = 1\Big) \cdot 1 + 0 \\
&= 1,
\end{aligned}$$

where F is the distribution function of the random variable $\sum_{j=1}^{\infty} W_j$.

(b) Two good resources for such simulation are the distr package in R, which is showcased by this tutorial. For Python 3 this tutorial uses the pyMC3 modules to provide a achieve a similar goal. We include R code for a single simulation, with parts taken from the tutorial indicated above.

```
library(distr)
library(coda)
library(latex2exp)

# Setup
set.seed(7)
n <- 10
alpha_vec <- c(0.1,1,10)
x_grid <- seq(-3, 3, by=0.05)
signif_level <- 0.05

# Sample observations
x_pts <- rnorm(n)

# Generate DKW Band
x_ecdf <- ecdf(x_pts)
x_ecdf_error <- sqrt(log(2 / signif_level) / (2 * n))
```

```
dkw.lb <- pmax(x_ecdf(x_grid) - x_ecdf_error, 0)
dkw.ub <- pmin(x_ecdf(x_grid) + x_ecdf_error, 1)

## BAYESIAN BANDS
# Functions to generate Bayesian Credible Bands
sample_cdf <- function(F_hat, n){
  F_hat@r(n) # F_hat is a S6 class object from the distr package
}

# Sampling from the prior distribution
sample_prior <- function(F0, alpha, n){
  cdf_sample <- sample_cdf(F0, n)
  v <- rbeta(n, 1, alpha) # See 5a) for definitions
  w <- c(v[1], rep(0, n-1))
  for(ii in 2:n){
    w[ii] <- v[ii]*cumprod(1-v)[ii-1] # See 5a) for definitions
  }
  function(m){
    sample(cdf_sample, m, prob=w, replace=T)
  }
}

# Sampling from the posterior distribution
sample_posterior <- function(F0,alpha,data){
  n <- length(data)
  F_hat <- DiscreteDistribution(data) #distr function for empirical CDF
  F_for_post <- n/(n+alpha)*F_hat+alpha/(n+alpha)*F0
  sample_prior(F_for_post,alpha+n,n)
}

# Now simulate for all different alphas
list_out_bayes <- list()
for(alpha in alpha_vec){
  iters <- 100
  m <- 1000
  F0 <- DiscreteDistribution(rnorm(m))

  y <- matrix(nrow=length(x_grid), ncol=iters)
  for(iter in 1:iters){
    F_post <- sample_posterior(F0, alpha, x_pts)
    y[,iter] <- ecdf(F_post(m))(x_grid)
  }

  mean_post_sim <- rowMeans(y) #Posterior Mean
  cred_int <- apply(y, 1, function(row) HPDinterval(as.mcmc(row),prob=signif_level))
      #obtains 95% credible interval
  list_out_bayes[[as.character(alpha)]] <- list('cred_int' = cred_int,
      'mean_post_sim'=mean_post_sim)
}


# Plot the results for each of the alpha
for (alpha_val in alpha_vec){
  plot(x_ecdf, xlim=c(-3, 3),
       main = TeX(sprintf("95%% DKW and Bayesian Credible Band ($\\alpha = %s$)",
           as.character(alpha_val)))) #ECDF
  lines(x_grid, dkw.lb, col="green") #DKW
  lines(x_grid, dkw.ub, col="green")

  points(x_grid, list_out_bayes[[as.character(alpha_val)]]$'mean_post_sim',
      type="l", col="red")
  points(x_grid, list_out_bayes[[as.character(alpha_val)]]$'cred_int'[2,], type='l',
      col='blue')
  points(x_grid, list_out_bayes[[as.character(alpha_val)]]$'cred_int'[1,], type='l',
      col='blue')
  curve(pnorm, xlim=c(-3, 3), add=TRUE, col="red", lwd=2)
  legend("topleft", lty="solid", legend=c("True", "DKW", "Kolmogorov", "Posterior
      Mean"),
        col=c("black", "green", "blue", "red"))
}
```

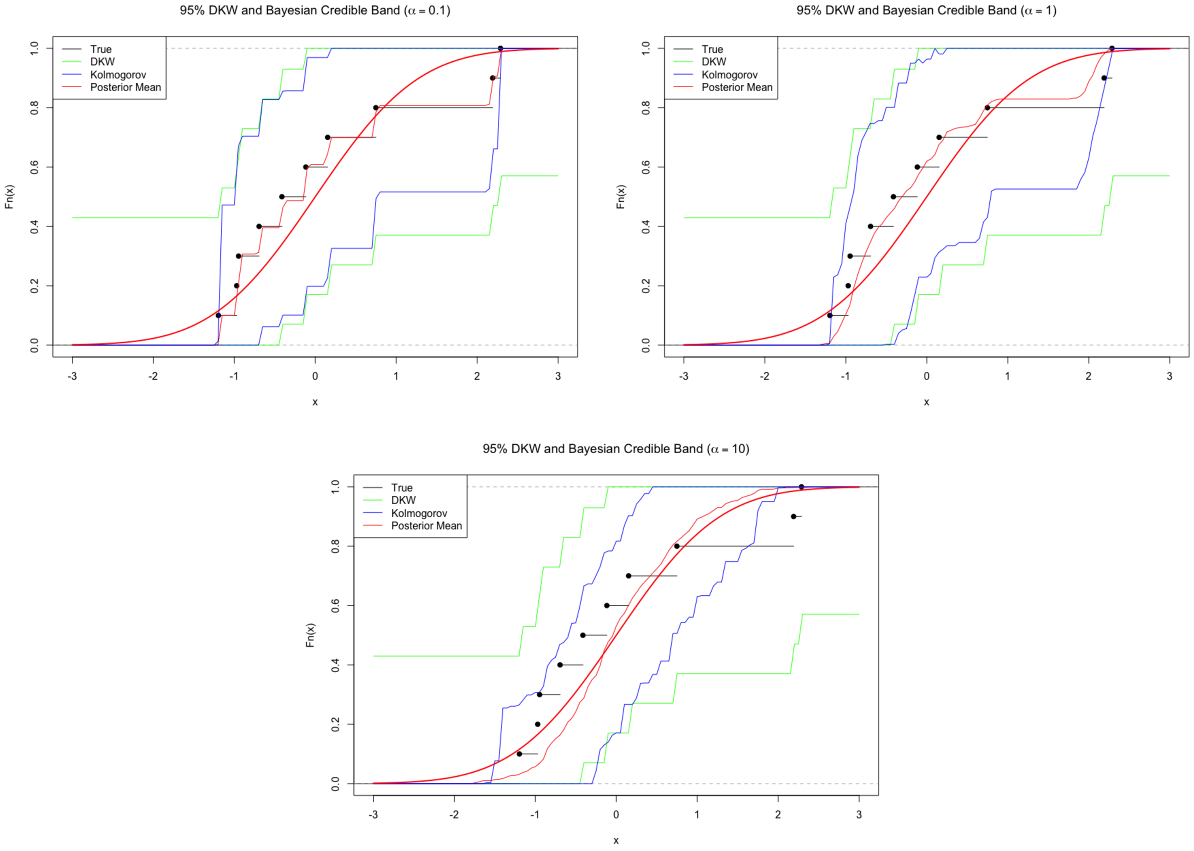


Figure 1: 95% DKW and Bayesian credible bands at different alpha levels: 0.1, 1, 10 from upper left to bottom center respectively

For $n = 1,000$ simulations, one should replicate the above code and consider whether the full empirical CDF is captured between the bands for both the Bayesian confidence and DKW bands.

Problem 6 [20 pts.]

In this question we consider a nonparametric Bayesian estimator and compare to the minimax estimator. For $i = 1,\ldots,n$ and $j = 1,2,\ldots$ let

$$X_{ij} = \theta_j + \epsilon_{ij}$$

where all the $\epsilon_{ij}'s$ are independent $N(0,1)$. The parameter is $\theta = (\theta_1,\theta_2,\ldots)$. Assume that $\sum_j \theta_j^2 < \infty$. Due to sufficiency, we can reduce the problem to the sample means. Thus let $Y_j = n^{-1}\sum_{i=1}^n X_{ij}$. So the model is $Y_j \sim N(\theta_j, 1/n)$ for $j = 1,2,3,\ldots$. We will put a prior π on θ as follows. We take each θ_j to be independent and we take $\theta_j \sim N(0,\tau_j^2)$.

(a) **(5 pts.)** Find the posterior for θ. Find the posterior mean $\widehat{\theta}$.

(b) **(7 pts.)** Suppose that $\sum_j \tau_j^2 < \infty$. Show that $\widehat{\theta}$ is consistent, that is, $\|\widehat{\theta}-\theta\|^2 \xrightarrow{P} 0$.

(c) **(8 pts.)**Now suppose that θ is in the Sobolev ball

$$\Theta = \left\{\theta = (\theta_1,\theta_2,\ldots) : \sum_j j^{2p}\theta_j^2 \le C^2\right\}$$

where $p > 1/2$. The minimax (for squared error loss) for this problem is $R_n \asymp n^{-2p/(2p+1)}$. Let $\tau_j^2 = (1/j)^{2r}$. Find r so that the posterior mean achieves the minimax rate.

Solution.

(a) By Theorem 6 (in the appendix) we have

$$\widehat{\theta}_j = \frac{nY_j\tau_j^2}{1+n\tau_j^2}.$$

(b) For any $\epsilon > 0$,

$$\begin{aligned}
P\Big(\|\widehat{\theta}-\theta\|^2 > \epsilon\Big) &\le \frac{\mathbb{E}\Big[\|\widehat{\theta}-\theta\|^2\Big]}{\epsilon} && \text{Markov's inequality}\\
&= \frac{1}{\epsilon}\sum_{j=1}^{\infty}\mathbb{E}\big[(\widehat{\theta}_j-\theta_j)^2\big]\\
&= \frac{1}{\epsilon}\left[\sum_{j=1}^{\infty}\Big(\mathbb{E}[\widehat{\theta}_j-\theta_j]\Big)^2 + \sum_{j=1}^{\infty}\mathrm{Var}(\widehat{\theta}_j)\right]\\
&= \frac{1}{\epsilon}\left[\sum_{j=1}^{\infty}\theta_j^2\frac{1}{(1+n\tau_j^2)^2} + \sum_{j=1}^{\infty}\frac{\tau_j^2}{1+n\tau_j^2}\right] && \text{Theorem 6}\\
&\to 0,
\end{aligned}$$

as $n \to \infty$, since $\sum_j \tau_j^2 < \infty$ and $\sum_j \theta_j^2 < \infty$.

(c) See Shen and Wasserman (2001).

Theorem 8 *Let $X_1, \dots, X_n \sim N(\mu, \sigma^2)$ where σ^2 is known. Let $\mu \sim N(a, b^2)$. Then,*

$$\mathbb{E}[\mu | X_1, \dots, X_n] = \frac{a\sigma^2 + n\overline{X}b^2}{\sigma^2 + nb^2}.$$

Proof.

$$f_{X^n}(x^n|\mu) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{1}{\sigma^2}(x_i - \mu)^2\} = (2\pi\sigma^2)^{-n/2} \exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu)^2\}$$

$$= (2\pi\sigma^2)^{-n/2} \exp\Big\{\frac{-1}{2\sigma^2}[(n-1)s^2 + n(\mu - \overline{X})^2]\Big\} = (2\pi\sigma^2)^{-n/2} \exp\Big\{\frac{-(n-1)s^2}{2\sigma^2}\Big\} \exp\Big\{\frac{-n(\mu - \overline{X})^2}{2\sigma^2}\Big\}$$

$$\propto \exp\Big\{\frac{-n(\mu - \overline{X})^2}{2\sigma^2}\Big\}.$$

$$\pi(\mu) = \frac{1}{\sqrt{2\pi b^2}} \exp\Big\{-\frac{1}{2b^2}(\mu - a)^2\Big\} \propto \exp\Big\{-\frac{1}{2b^2}(\mu - a)^2\Big\}.$$

Hence,

$$\pi(\mu|X^n) \propto f_{X^n}(x^n|\mu)\pi(\mu) \propto \exp\Big\{\frac{-n(\mu - \overline{X})^2}{2\sigma^2} - \frac{1}{2b^2}(\mu - a)^2\Big\}$$

$$= \exp\Big\{\frac{-n\mu^2 + 2n\mu\overline{X} - n\overline{X}^2}{2\sigma^2} + \frac{-\mu^2 + 2\mu a - a^2}{2b^2}\Big\}$$

$$= \exp\Big\{\mu^2\Big(\frac{-1}{2b^2} - \frac{n}{2\sigma^2}\Big) - 2\mu\Big(-\frac{a}{2b^2} - \frac{n\overline{X}}{2\sigma^2}\Big) - \Big(\frac{a^2}{2b^2} + \frac{n\overline{X}^2}{2\sigma^2}\Big)\Big\}$$

For simplicity, let

$$U = \Big(\frac{-1}{2b^2} - \frac{n}{2\sigma^2}\Big) \quad \text{and} \quad V = \Big(-\frac{a}{2b^2} - \frac{n\overline{X}}{2\sigma^2}\Big).$$

Then

$$\pi(\mu|X^n) \propto \exp\{U\mu^2 - 2V\mu\} = \exp\Big\{U\Big(\mu^2 - 2\mu\frac{V}{U} + \frac{V^2}{U^2}\Big) - \frac{V^2}{U}\Big\}$$

$$\propto \exp\Big\{U(\mu - \frac{V}{U})^2\Big\} = \exp\Big\{\frac{-1}{2(1/\sqrt{-2U})^2}(\mu - \frac{V}{U})^2\Big\} \propto N\Big(\frac{V}{U}, \frac{-1}{2U}\Big).$$

Therefore the mean of the posterior is,

$$\widehat{\mu} = \mathbb{E}[\mu|X^n] = \frac{V}{U} = \frac{-a\sigma^2 - nb^2\overline{X}}{-\sigma^2 - nb^2} = \boxed{\frac{a\sigma^2 + n\overline{X}b^2}{\sigma^2 + nb^2}}.$$