Homework 4
Due Friday April 19 3:00 pm
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1. Consider the directed graph with vertices $V = \{X_1, X_2, X_3, X_4, X_5\}$ and edge set $E = \{(1,3), (2,3), (3,4), (3,5)\}$.

   (a) List all the independence statements implied by this graph.

   (b) Find the causal distribution $p(x_4 | \text{set } x_3 = s)$.

   (c) Find the implied undirected graph for these random variables. Which independence statements get lost in the undirected graph (if any)?

2. Let $d \geq 2$, and let $X_1, \ldots, X_n \sim P$ where $X_i = (X_i(1), \ldots, X_i(d)) \in \mathbb{R}^d$. Assume that the coordinates of X_i are independent. Further, assume that $X_i(j) \sim \text{Bernoulli}(p_j)$ where $0 < c \leq p_j \leq C < 1$. Let $\mathcal{P}$ be all such distributions. Let

   $$R_n = \inf_{\widehat{p}} \sup_{P \in \mathcal{P}} \mathbb{E}_P ||\widehat{p} - p||_\infty.$$

   Find lower and upper bounds on the minimax risk.

3. Let $\{p_\theta : \theta \in \Theta\}$ where $\Theta \subset \mathbb{R}$ be a parametric model. Suppose that the model satisfies the usual regularity conditions. In particular, the Fisher information $I(\theta)$ is positive and smooth and the mle has the usual nice properties. Let the loss function be $L(\widehat{\theta}, \theta) = H(p_{\widehat{\theta}}, p_\theta)$ where H denotes Hellinger distance. Find the minimax rate.

4. Let $Y = (Y_1, \ldots, Y_d) \sim N(\theta, I)$ where $\theta = (\theta_1, \ldots, \theta_d)$. Assume that $\theta \in \Theta = \{\theta \in \mathbb{R}^d : ||\theta||_0 \leq 1\}$. Let

   $$R_d = \inf_{\widehat{\theta}} \sup_{\theta \in \Theta} \mathbb{E}_\theta ||\widehat{\theta} - \theta||^2.$$

   Show that $c \log d \leq R_d \leq C \log d$ for some constants c and C.

5. Let $X_1, \ldots, X_n \sim F$ where F is some distribution on $\mathbb{R}$. Suppose we put a Dirichlet process prior on F:

   $$F \sim \text{DP}(\alpha, F_0).$$

   (a) Recall the stick-breaking construction. Show that $\mathbb{E}(\sum_{j=1}^\infty W_j) = 1$.

   (b) Simulate $n = 10$ data points from a $N(0,1)$. Try three values of α: namely, $\alpha = .1$, $\alpha = 1$ and $\alpha = 10$. Compute the 95 percent Bayesian confidence band and the 95 percent DKW band. Plot the results for one example. Now repeat the simulation 1,000 times and report the coverage probability for each confidence band.

6. For $i = 1, \ldots, n$ and $j = 1, 2, \ldots$ let

   $$X_{ij} = \theta_j + \epsilon_{ij}$$

where all the $\epsilon_{ij}'s$ are independent N(0,1). The parameter is $\theta = (\theta_1, \theta_2, \ldots,)$. Assume that $\sum_j \theta_j^2 < \infty$. Due to sufficiency, we can reduce the problem to the sample means. Thus let $Y_j = n^{-1} \sum_{i=1}^n X_{ij}$. So the model is $Y_j \sim N(\theta_j, 1/n)$ for $j = 1, 2, 3 \ldots$ We will put a prior π on θ as follows. We take each θ_j to be independent and we take $\theta_j \sim N(0, \tau_j^2)$.

(a) Find the posterior for θ. Find the posterior mean $\widehat{\theta}$.

(b) Suppose that $\sum_j \tau_j^2 < \infty$. Show that $\widehat{\theta}$ is consistent, that is, $||\widehat{\theta} - \theta||^2 \xrightarrow{P} 0$.

(c) Now suppose that θ is in the Sobolev ball

$$\Theta = \left\{ \theta = (\theta_1, \theta_2, \ldots) : \sum_j j^{2p} \theta_j^2 \le C^2 \right\}$$

where $p > 1/2$. The minimax (for squared error loss) for this problem is $R_n \asymp n^{-2p/(2p+1)}$. Let $\tau_j^2 = (1/j)^{2r}$. Find r so that the posterior mean achieves the minimax rate.