Practice Final Exam

36-325/725 Fall, 2001

(1) Let X have probability density function

$$f_X(x) = \begin{cases} \frac{1}{2} & 0 < x < 1 \\ \frac{1}{2} & 3 < x < 4 \\ 0 & \text{otherwise.} \end{cases}$$

Find the cumulative distribution function of X.

(2) Let $X_1, \ldots, X_n$ be random variables and suppose that $Var(X_i) = \sigma^2 < \infty$ for $i = 1, \ldots, n$. Also assume that $\rho(X_i, X_j) = \rho$ for all i, j such that $i \neq j$. Show that $\rho \geq -1/(n-1)$.

(3) Let $X_1, \ldots, X_n$ be i.i.d. observations from a Normal distribution with mean θ and variance 1. Define

$$Y_i = \begin{cases} 1 & \text{if } X_i > 0 \\ -1 & X_i \leq 0. \end{cases}$$

Let $\psi = E(Y_1)$.

(3a) Find the maximum likelihood estimate $\hat{\psi}$ of ψ.

(3b) Find an approximate 95 per cent confidence interval for ψ.

(3c) Define

$$\tilde{\psi} = \frac{1}{n} \sum_{i=1}^{n} Y_i.$$

Compute the asymptotic relative efficiency of $\tilde{\psi}$ to $\hat{\psi}$.

(4) Let $X_1, \ldots, X_n$ be iid with a Uniform $(-\theta, \theta)$ distribution where $\theta > 0$.

(4a) Find the likelihood function for θ.

(4b) Find the maximum likelihood estimator $\hat{\theta}$.

(4c) Find the exact distribution of $\hat{\theta}$.

(5) Let $X_1, \ldots, X_n$ be i.i.d. observations from a Bernoulli (p) distribution.

(5a) Find the maximum likelihood estimator for the parameter p.

(5b) Find the Fisher information.

(5c) Based on the large sample theory for maximum likelihood estimators, what is the (approximate) distribution of $\hat{p}$?

(5d) Find the MSE (mean squared error) of $\hat{p}$.

(6) Suppose that

$$Y_i = \beta x_i + \epsilon_i$$

where $\epsilon_1, \ldots, \epsilon_n$ are independent, $E(\epsilon_i) = 0$ and $Var(\epsilon_i) = \sigma^2$.

(6a) Find the least squares estimate $\hat{\beta}$ of β.

(6b) Prove that $\hat{\beta}$ it is unbiased as long as at least one $x_i \neq 0$.

(6c) Show that $\hat{\beta}$ is a consistent estimator. You will need to place a condition on the x_i's. Be explicit about the condition that you need for consistency.

(**7**) (15 points) Let X have density

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$Y = \begin{cases} 0 & \text{if } X \leq 1 \\ X & \text{otherwise.} \end{cases}$$

(7a) Find the cumulative distribution function for Y.

(7b) Find $E(Y)$.

(7c) Find $E(Y|X)$.

(**8**) (20 points) Let $X_1, \ldots, X_n$ be iid with a Uniform $(\theta, 2\theta)$ distribution where $\theta > 0$.

(8a) Find the maximum likelihood estimator $\hat{\theta}$.

(8b) Find the exact distribution of $\hat{\theta}$.

(8c) Find $E(\hat{\theta})$ for the case $n = 2$.

(9) Let $X_1, \ldots, X_n$ be i.i.d. observations from a Poisson (λ) distribution.

(i) Show that the maximum likelihood estimate is $\hat{\lambda} = \overline{X} = n^{-1} \sum_{i=1}^{n} X_i$.

(ii) Show that the Fisher information is $I(\lambda) = 1/\lambda$.

(iii) Based on the large sample theory for maximum likelihood estimators, what is the (approximate) distribution of $\hat{\lambda}$?

(iv) What is the exact distribution of $\hat{\lambda}$? (i.e. what is the probability function for $\hat{\lambda}$?)

Hint: Compute $Pr(\hat{\lambda} \leq c)$ for any real number c.

(v) Let $\psi = Pr(X_1 = 0)$. Find an approximate 95 per cent confidence interval for ψ.

(10) Let $X_1, \ldots, X_n$ be i.i.d. observations from a Poisson (λ) distribution. Consider the prior probability density function $f(\lambda) = 1/\sqrt{\lambda}$.

(i) Find the posterior distribution for λ.

(ii) Let $\overline{\lambda} = E(\lambda | X_1, \ldots, X_n)$ be the Bayes estimator. Show that

$$\overline{\lambda} = \frac{\sum_i X_i + \frac{1}{2}}{n}.$$