Homework 11
Due Friday Nov 30

(1) Consider problem 3 from Homework 10. Suppose the data are:

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3.23 -2.50  1.88 -0.68  4.43  0.17
1.03 -0.07 -0.01  0.76  1.76  3.18
0.33 -0.31  0.30 -0.61  1.52  5.43
1.54  2.28  0.42  2.33 -1.03  4.00
0.39
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Find the mle $\hat{\tau}$. Find the standard error using the delta method. Find the standard error using the parametric bootstrap.

(2) Let $X_1, ..., X_n$ $\text{Normal}(\mu, 1)$. Let $\theta = e^{\mu}$ and let $\widehat{\theta} = e^{\overline{X}}$ be the mle. Create a data set (using $\mu = 5$) consisting of n=100 observations.

(2a) Use the delta-method to get the se and 95 percent confidence interval for θ. Use the parametric bootstrap to get the se and 95 percent confidence interval for θ. Use the nonparametric bootstrap to get the se and 95 percent confidence interval for θ. Compare your answers.

(2b) Plot a histogram of the bootstrap replications for the parametric and nonparametric bootstraps. These are estimates of the distribution of $\hat{\theta}$. The delta method also gives an approximation to this distribution namely, Normal($\hat{\theta}, se^2$). Compare these to the true sampling distribution of $\hat{\theta}$. Which approximation, parametric bootstrap, bootstrap, or delta method is closer to the true distribution?

(3) Let $X_1, ..., X_n$ $\text{Unif}(0, \theta)$. The mle is $\hat{\theta} = X_{max} = \max\{X_1, ..., X_n\}$. Generate a data set of size 50 with $\theta = 1$.

(3a) Find the distribution of $\hat{\theta}$. Compare the true distribution of $\hat{\theta}$ to the histograms from the parametric and nonparametric bootstraps.

(3b) This is a case where the nonparametric bootstrap does very poorly. In fact, we can prove that this is the case. Show that, for the parametric bootstrap $P(\hat{\theta}^* = \hat{\theta}) = 0$ but for the nonparametric bootstrap $P(\hat{\theta}^* = \hat{\theta}) \approx .632$. Hint: show that, $P(\hat{\theta}^* = \hat{\theta}) = 1 - (1 - (1/n))^n$ then take the limit as n gets large.

(4) Suppose that 50 people are given a placebo and 50 are given a new treatment. 30 placebo patients show improvement while 40 treated patients show im-

provement. Let $\tau = p_2 - p_1$ where p_2 is the probability of improving under treatment and p_1 is the probability of improving under placebo.

(4a) Find the mle of τ. Find the standard error and 90 per cent confidence interval using the delta method.

Note: Let $X_1 = 30$ and $X_2 = 40$. Treat (X_1, X_2) as a single observation with probability function $f(x_1, x_2;\ p_1, p_2, n_1, n_2) = f(x_1;\ p_1, n_1) f(x_2; p_2, n_2)$ where $f(x;\ p, n)$ is a Binomial. Compute the Fisher information from $f(x_1, x_2;\ p_1, p_2, n_1, n_2)$; there is no need to divide the resulting standard error by $\sqrt{n}$.

(4b) Find the standard error and 90 per cent confidence interval using the bootstrap.

(4c) Use the prior $f(p_1, p_2) = 1$. Use simulation to find the posterior mean and posterior 90 per cent interval for τ.

(4d) Let

$$\psi = \log\left(\left(\frac{p_1}{1-p_1}\right) \div \left(\frac{p_2}{1-p_2}\right)\right)$$

be the log-odds ratio. Note that $\psi = 0$ if $p_1 = p_2$. Find the mle of ψ. Use the delta method to find a 90 per cent confidence interval for ψ.

(4e) Use simulation to find the posterior mean and posterior 90 per cent interval for ψ.

(5) Let $X_1, \ldots, X_n \sim N(\theta, \sigma^2)$. Assume that σ is known. Use the prior $\theta \sim N(a, b^2)$. Show that $\theta | x^n \sim N(\overline{\theta}, \tau^2)$ where

$$\overline{\theta} = w\overline{X} + (1-w)a,$$

$$w = \frac{\frac{1}{se^2}}{\frac{1}{se^2} + \frac{1}{b^2}} \quad \text{and} \quad \frac{1}{\tau^2} = \frac{1}{se^2} + \frac{1}{b^2}$$

and $se = \sigma/\sqrt{n}$ is the standard error of the mle $\overline{X}$.

(6) Consider the Bernoulli(p) observations

0 1 0 1 0 0 0 0 0 0

Plot the posterior for p using these priors: Beta(1/2,1/2), Beta(1,1), Beta(10,10), Beta(100,100).

(7) Let $X_1, \ldots, X_n \sim \text{Poisson}(\lambda)$.

(7a) Let $\lambda \sim \mathrm{Gamma}(\alpha, \beta)$ be the prior. Show that the posterior is also a Gamma. Find the posterior mean.

(7b) Find the Jeffreys' prior. Find the posterior.