Homework 3
36-325/725
due Friday Sept 14

(1) Chapter 2.1 Problem # 6.

A box contains three cards. One card is red on both sides, one card is green on both sides, and one card is red on one side and green on the other. Select a card at random and observe the color on one side. If this side is green, what is the probability that the other side is green?

Hint: List the sample space carefully.

(2) Chapter 2.1 Problems # 11 and 12

The probability that a child has blue eyes is 1/4. Assume independence between children. Consider family with 5 children.

(a) If it is known that at least one child has blue eyes, what is the probability that at least three children have blue eyes?

(b) If it is known that the youngest child has blue eyes, what is the probability that at least three children have blue eyes?

(3) Chapter 2.2 Problem # 4

Suppose k events form a partition of the sample space S, i.e. they are disjoint and $\cup_{i=1}^{k} A_i = S$. Assume that $P(B) > 0$. Prove that if $P(A_1|B) < P(A_1)$ then $P(A_i|B) > P(A_i)$ for some $i = 2, \ldots, k$.

(4) Chapter 2.2 Problem # 8

In a certain city, 30 percent of the people are conservatives, 50 percent of the people are liberals, and 20 percent of the people are independents. Suppose that 65 percent of the conservatives voted, 82 percent of the liberals voted, and 50 percent of the independents voted. We select a person at random and learn that she did not vote. What is the probability that she is a liberal?

(5) Chapter 2.2 Problems # 10 and 11.

A box contains 5 coins and each has a different probability of showing heads. Let $p_1, \ldots, p_5$ denote the probability of heads on each coin. Suppose that

$$p_1 = 0, p_2 = 1/4, p_3 = 1/2, p_4 = 3/4 \text{ and } p_5 = 1.$$

Let H denote "heads is obtained" and let C_i denote the event that coin i is selected.

(a) Select a coin at random and toss it. Suppose a head is obtained. What is the posterior probability that coin i was selected ($i = 1, \ldots, 5$)? In other words, find $P(C_i|H)$ for $i = 1, \ldots, 5$.

(b) Toss the coin again. What is the probability of another head? In other words find $P(H_2|H_1)$ where $H_j =$ "heads on toss j."

Now suppose that the experiment was carried out as follows. We select a coin at random and toss it until a head is obtained.

(c) Find $P(C_i|B_4)$ where $B_4 =$ "first head is obtained on toss 4."

(6) Here we will get some experience simulating conditional probabilities.

(a) Consider tossing a fair die. Let $A = \{2, 4, 6\}$ and $B = \{1, 2, 3, 4\}$. Then, $P(A) = 1/2$, $P(B) = 2/3$ and $P(AB) = 1/3$. Since $P(AB) = P(A)P(B)$, the events A and B are independent. Let's verify this numerically.

```
n <- 10000
p <- rep(1/6,6)
tosses <- sample(1:6,size=n,replace=T,prob=p)
help(sample)
par(mfrow=c(2,2))
plot(1:6,p,type='h',lwd=3,ylim=c(0,1))
z <- tabulate(tosses,nbins=6)
z <- z/n
plot(1:6,z,type='h',lwd=3,ylim=c(0,1))

A <- tosses[ (tosses == 2) | (tosses == 4) | (tosses == 6) ]
### the vertical bar means 'or'
pA <- length(A)/n
B <- tosses[ (tosses == 1) | (tosses == 2) |
               (tosses == 3) | (tosses == 4) ]
pB <- length(B)/n
AB <- tosses[  (tosses == 2) |  (tosses == 4) ]
```

```
pAB <- length(AB)/n
pA
pB
pAB
pA*pB
pA.given.B <- pAB/pB
pA.given.B
```

Now create two events that are not independent. Compute $P(A), P(B), P(AB)$ and $P(AB)$. Compare the calculated values to their theoretical values. Report your results. Now repeat using n = 10. Report your results. Interpret.

(7) Simulate question 5a. Plot the simulated values for $P(C_i|H)$, $i = 1, \ldots, 5$. Do this for $n = 10$ and $n = 10,000$. Compare to the theoretical values. Here is some code to get you started:

```
p <- c(0, 1/4, 1/2, 3/4, 1)
### Here is a single experiment
### select a coin at random:
coin <- sample(1:5,size=1,replace=T,prob=rep(1/5,5))
### toss the coin:
A <- rbinom(1,1,p[coin])
print( c(coin,A) )

### to repeat this n times:
coin <- sample(1:5,size=n,replace=T,prob=rep(1/5,5))
A <- rbinom(n,1,p[coin])
### to make sure you understand this first use n = 10
### and print out coin and p[coin]
```