**Homework 4**
**36-325/725**
**due Friday Sept 21**

(1a) Let $X$ be such that $P(X = 2) = P(X = 3) = 1/10$ and $P(X = 5) = 8/10$. Draw a graph of the cdf $F$. Use $F$ to find $P(2 < X \leq 4.8)$ and $P(2 \leq X \leq 4.8)$.

(1b) Let $X$ be a discrete random variable. Prove that $P(X = x) = F(x^+) - F(x^-)$.

(2) Let $X$ have cdf $F$. Let $X^+ = \max\{0, X\}$. Find the cdf of $X^+$.

(3) Let $X$ be a random variable such that $P(X > 0) = 1$. Suppose that $X$ has continuous, differentiable cdf $F$, pdf $f$ and suppose that $F(x) < 1$ for all $x < \infty$. Define $H(x) = -\log(1 - F(x))$ and let $h(x) = H'(x) = f(x)/(1 - F(x))$. The function $h$ is called the hazard function. Think of $X$ as a lifetime; for example, the lifetime of a transistor or a light bulb or a person.

(3a) Show that

$$h(x) = \lim_{h\downarrow 0} \frac{P(X \leq x + h | X > x)}{h}.$$

Use this result to give an interpretation of $h$.

(3b) Show that, for all $x, y$,

$$P(X > x + y | X > x) = \exp\left\{- \int_x^{x+y} h(u)du\right\}.$$

(3c) Suppose that $X \sim \text{Exp}(\beta)$. Show that $h(x) = 1/\beta$. Interpret this result.

(4) Let $X \sim N(0, 1)$ and let $Y = e^X$.

(4a) Find the pdf for $Y$. Plot it in R.

(4b) Generate a vector $x$ consisting of 10,000 random standard Normals in $R$. Then:

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y <- e^x
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Draw a histogram of $y$ and compare it to the pdf from (3a).

(5) Let $(X, Y)$ be uniformly distributed on the unit disc $\{(x, y) : x^2 + y^2 \leq 1\}$. Let $R = \sqrt{X^2 + Y^2}$. Find the cdf and pdf of $R$. Hint: $F_R(r) = P(R \leq r) = P(\sqrt{X^2 + Y^2} \leq r) = P(X^2 + Y^2 \leq r^2)$. Now use geometric reasoning.

(6) Let $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$ and assume that $X$ and $Y$ are independent. Show that the distribution of $X$ given that $X + Y = n$ is $\text{Binomial}(n, \pi)$ where $\pi = \lambda/(\lambda + \mu)$.

Hint 1: You may use the following fact: If $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$, and $X$ and $Y$ are independent, then $X + Y \sim \text{Poisson}(\mu + \lambda)$.

Hint 2: Note that $\{X = x,\ X + Y = n\} = \{X = x,\ Y = n - x\}$.

(7) Let

$$f_{X,Y}(x, y) = \begin{cases} c(x + y^2) & 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $P\left(X < \frac{1}{2} \mid Y = \frac{1}{2}\right)$.

The following are practice problems. Do not hand them in.

(8) Do the following questions two ways: using $R$ and using the tables at the back of DeGroot.

(8a) Let $X \sim N(3, 16)$. Find $P(X < 7)$.

(8b) Let $X \sim N(3, 16)$. Find $P(X > -2)$.

(8c) Let $X \sim N(3, 16)$. Find $x$ such that $P(X > x) = .05$.

(8d) Let $X \sim N(3, 16)$. Find $P(0 \leq X < 4)$.
(8e) Let $X \sim N(3, 16)$. Find $x$ such that $P(|X| > |x|) = .05$.
(8f) Let $X \sim \mathrm{Poisson}(10)$. Find $P(9 \leq X \leq 12)$.