Practice Test 2
36-325/725
Fall 2001

(1) Suppose that there is a computer cluster with 24 computers and that a class has 25 students. The probability that any one student will show up tomorrow and want to use a computer in the cluster is .9. Suppose that students are independent. Use the central limit theorem to approximate the probability that there will be more students than computers i.e. that $P(X > 24)$.

(2) Suppose we have a computer program consisting of $n = 100$ pages of code. Let X_i be the number of errors on the i^{th} page of code. Suppose that the $X_i's$ are Poisson with mean 1 and that they are independent. Let $Y = \sum_{i=1}^{n} X_i$ be the total number of errors. Use the central limit theorem to approximate $P(Y < 90)$.

(3) Define (i) "X_n converges to X in probability" and (ii) "X_n converges to X in distribution". When does one type of convergence imply the other? (No need to supply proofs.)

(4) Suppose that $P(X = 1) = P(X = -1) = 1/2$. Define

$$X_n = \begin{cases} X & \text{with probability } 1 - \frac{1}{n} \\ e^n & \text{with probability } \frac{1}{n}. \end{cases}$$

Does X_n converge to X in probability? Does X_n converge to X in distribution? Does $E(X - X_n)^2$ converge to 0?

(5) Let $Z \sim N(0,1)$. Let $t > 0$.
(5a) Show that, for any $k > 0$,

$$P(|Z| > t) \leq \frac{E|Z|^k}{t^k}.$$

(5b) Show that

$$P(|Z| > t) \leq \left\{ \frac{2}{\pi} \right\}^{1/2} \frac{e^{-t^2/2}}{t}.$$

Hint 1: Note that $P(|Z| > t) = 2P(Z > t)$.

Hint 2: Write out what $P(Z > t)$ means and note that $x/t > 1$ whenever $x > t$.

(6) Suppose that $X_n \sim N(0, 1/n)$ and let X be a random variable with distribution $F(x) = 0$ if $x < 0$ and $F(x) = 1$ if $x \geq 0$. Does X_n converge to X in probability? (Prove or disprove). Does X_n converge to X in distribution? (Prove or disprove).

(7) Let $X, X_1, X_2, X_3, \ldots$ be random variables that are positive and integer valued. Show that $X_n \xrightarrow{d} X$ if and only if

$$\lim_{n \to \infty} P(X_n = k) = P(X = k)$$

for every integer k.

(8) Let $Z_1, Z_2, \ldots$ be i.i.d., random variables with density f. Suppose that $P(Z_i > 0) = 1$ and that $\lambda = \lim_{x \downarrow 0} f(x) > 0$. Let

$$X_n = n \min\{Z_1, \ldots, Z_n\}.$$

Show that $X_n \xrightarrow{d} Z$ where Z has an exponential distribution with mean $1/\lambda$.

(9) Let X_n be a random variable with probability mass function

$$p_n(x) = \begin{cases} \frac{1}{2} & \text{if } x = -\left(\frac{1}{2}\right)^n \\ \frac{1}{2} & \text{if } x = \left(\frac{1}{2}\right)^n \\ 0 & \text{otherwise.} \end{cases}$$

Let $p(x) = \lim_{n \to \infty} p_n(x)$. Is $p(x)$ a probability function? Does X_n converge in distribution to some random variable?