Homework 1: Solutions

1. Let $B_n = \cup_{i=n}^{\infty} A_i$ and $C_n = \cap_{i=n}^{\infty} A_i$.

(1a) $s \in C_n \implies s \in \cap_{i=n}^{\infty} A_i \implies s \in \cap_{i=n+1}^{\infty} A_i \implies s \in C_{n+1}$. Therefore, $C_n \subset C_{n+1}$.

$s \in B_n \implies s \in \cup_{i=n}^{\infty} A_i \implies s \in A_{n-1} \cup (\cup_{i=n}^{\infty} A_i) \implies s \in \cup_{i=n-1}^{\infty} A_i \implies s \in B_{n-1}$. Therefore, $B_n \subset B_{n-1}$.

(1b) If $s \in A_n$ i.o. then $s \in \cup_{i=n}^{\infty} A_i$ for all $n \implies s \in B_n$ for all $n \implies s \in \cap_{n=1}^{\infty} B_n$.

If $s \notin A_n$ i.o. then there exists n such that $s \notin A_j$ for all $j > n \implies s \notin \cup_{i=n+1}^{\infty} \implies s \notin B_{n+1} \implies s \notin \cap_{n=1}^{\infty} B_n$. Thus, $s \in \cap_{n=1}^{\infty} B_n \implies$ that $s \in A_n$i.o..

(1c) $s \in A_j$ ult. $\implies s \in A_j$ for all $j \geq n$ (for some n) $\implies s \in \cap_{j=n}^{\infty} A_j = C_n \implies s \in \cup_{n=1}^{\infty} C_n$.

$s \in \cup_{n=1}^{\infty} C_n \implies s \in C_n$ for some $n \implies s \in \cap_{j=n}^{\infty} A_j \implies s \in A_j$ for all $j \geq n \implies s \in A_j$ ult.

2. $s \in (\cup_i A_i)^c$ iff $s \notin \cup_i A_i$ iff $s \notin A_i$ for all i iff $s \in A_i^c$ for all i iff $s \in \cap_i A_i^c$.

$s \in (\cap_i A_i)^c$ iff $s \notin \cap_i A_i$ iff $s \notin A_i$ for some i iff $s \in A_i^c$ for some i iff $s \in \cup_i A_i^c$.

3. Suppose there exists a P that is uniform on S. Then there is a constant c such that $c = P(\{s_i\})$ for all i. If $c = 0$ then $1 = P(S) = \sum_i P(\{s_i\}) = 0$ which is a contradiction. If $c > 0$ then $1 = P(S) = \sum_i P(\{s_i\}) = \infty$ which is a contradiction.

4. Let $B_n = A_n - \cup_{i=1}^{n-1} A_i$. Consider $j < n$. If $s \in B_j$ then $s \in A_j$. If $s \in B_n$ then $s \notin \cup_{i=1}^{n-1} A_i$ and hence $s \notin A_j$. Thus, $B_j \cap B_n = \emptyset$.

Next we claim that $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} B_i$. Let $s \in \cup_{i=1}^{\infty} A_i$. Then $s \in A_n$ for at least one n. Let n_0 be the smallest such n. Then $s \in A_{n_0}$, but for $j < n_0$, $s \notin A_j$. Thus, $s \in B_n$ and therefore $s \in \cup_{i=1}^{\infty} B_i$. This shows that

$\cup_{i=1}^{\infty} A_i \subset \cup_{i=1}^{\infty} B_i$. Now let $s \in \cup_{i=1}^{\infty} B_i$. Then $s \in B_n$ for some n. Hence, $s \in A_n$ and hence $s \in \cup_{i=1}^{\infty} A_i$. This shows that $\cup_{i=1}^{\infty} B_i \subset \cup_{i=1}^{\infty} A_i$. Therefore $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} B_i$.

Also note that $B_n \subset A_n$ and so $P(B_n) \leq P(A_n)$. Finally,

$$\begin{aligned} P\left(\cup_{i=1}^{\infty} A_i\right) &= P\left(\cup_{i=1}^{\infty} B_i\right) && \text{since } \cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} B_i \\ &= \sum_i P(B_i) && \text{since the } B_i \text{ are disjoint} \\ &\leq \sum_i P(A_i) && \text{since } P(B_i) \leq P(A_i). \end{aligned}$$

5. From question 4 we get that $P(\cup_i A_i^c) \leq \sum_i P(A_i^c) = 0$ since $P(A_i^c) = 1 - P(A_i) = 0$. Hence, $1 - P(\cup_i A_i^c) \geq 1$. Now,

$$\begin{aligned} 1 &\geq P(\cap_i A_i) \\ &= 1 - P\left(\cap_i A_i\right)^c \\ &= 1 - P\left(\cup_i A_i^c\right) \\ &\geq 1. \end{aligned}$$

Therefore, $1 = P(\cap_i A_i)$.