Solutions for Homework 12

(1) Let $Q = \sum_i (Y_i - (\beta_0 + \beta_1 x_i))^2$. Then

$$\begin{aligned}
\frac{\partial Q}{\partial \beta_0} &= -2 \sum_i (Y_i - (\beta_0 + \beta_1 x_i)) \\
\frac{\partial Q}{\partial \beta_1} &= -2 \sum_i x_i (Y_i - (\beta_0 + \beta_1 x_i)).
\end{aligned}$$

The solution follows by setting these equal to 0.

(2)

$$\frac{\sum_i (x_i - \overline{x})(Y_i - \overline{Y})}{\sum_i (x_i - \overline{x})^2} = \frac{\sum_i (x_i - \overline{x}) Y_i}{\sum_i (x_i - \overline{x})^2} - \frac{\overline{Y} \sum_i (x_i - \overline{x})}{\sum_i (x_i - \overline{x})^2} = \frac{\sum_i (x_i - \overline{x}) Y_i}{\sum_i (x_i - \overline{x})^2}$$

since $\sum_i (x_i - \overline{x}) = 0$. The other one is similar.

(3) The log-density is

$$\ell = -n \log \sigma - \frac{1}{2\sigma^2} \sum_i (Y_i - \mu_i)^2$$

where $\mu_i = \beta_0 + \beta_1 x_i$. The second derivatives are

$$\begin{aligned}
\frac{\partial^2 \ell}{\partial \beta_0^2} &= -\frac{n}{\sigma^2} \\
\frac{\partial^2 \ell}{\partial \beta_1^2} &= -\frac{\sum_i x_i^2}{\sigma^2} \\
\frac{\partial^2 \ell}{\partial \sigma^2} &= \frac{n}{\sigma^2} - \frac{3}{\sigma^4} \sum_i (Y_i - \mu_i)^2 \\
\frac{\partial^2 \ell}{\partial \beta_0 \partial \beta_1} &= -\frac{\sum_i x_i}{\sigma^2} \\
\frac{\partial^2 \ell}{\partial \beta_0 \partial \sigma} &= -\frac{2}{\sigma^3} \sum_i (Y_i - \mu_i) \\
\frac{\partial^2 \ell}{\partial \beta_1 \partial \sigma} &= -\frac{2}{\sigma^3} \sum_i x_i (Y_i - \mu_i).
\end{aligned}$$

Take minus one times the expected value to get

$$I(\beta_0, \beta_1, \sigma) = \begin{bmatrix} \frac{n}{\sigma^2} & \frac{\sum_i x_i}{\sigma^2} & 0 \\ \frac{\sum_i x_i}{\sigma^2} & \frac{\sum_i x_i^2}{\sigma^2} & 0 \\ 0 & & \frac{2n}{\sigma^2} \end{bmatrix}.$$

Now we need the inverse

$$J = I^{-1} = \begin{bmatrix} \frac{\sigma^2 \sum_i x_i^2}{n \sum_i x_i^2 - (\sum_i x_i)^2} & -\frac{\sigma^2 \sum_i x_i}{n \sum_i x_i^2 - (\sum_i x_i)^2} & 0 \\ -\frac{\sigma^2 \sum_i x_i}{n \sum_i x_i^2 - (\sum_i x_i)^2} & \frac{n\sigma^2}{n \sum_i x_i^2 - (\sum_i x_i)^2} & 0 \\ 0 & 0 & \frac{\sigma^2}{2n} \end{bmatrix}.$$

Now, the approximate variance of $\hat{\beta}_1$ is

$$J(2,2) = \frac{\sigma^2}{\sum_i (x_i - \overline{x})^2}.$$

This is the same as the exact expression for the variance.