Solutions to Homework 2

(1) Define B_n as in the hint. Consider $j \neq i$. Without loss of generality, assume $j < i$. If $s \in B_j$ then $s \in A_j$ so $s \notin B_i$. If $s \in B_i$ then $s \in A_j^c$ so $s \notin B_j$. Hence $B_i \cap B_j = \emptyset$. If $s \in \cup_{i=1}^{\infty} A_i$ then $s \in A_n$ for some n. Hence, $s \in B_n$ and thus $s \in \cup_{i=1}^{\infty} B_i$. This shows that $\cup_{i=1}^{\infty} A_i \subset \cup_{i=1}^{\infty} B_i$. If $s \in \cup_{i=1}^{\infty} B_i$ then $s \in B_n$ then for some n which implies that $s \in A_n$ and so $s \in \cup_{i=1}^{\infty} A_i$. Therefore, $\cup_{i=1}^{\infty} B_i \subset \cup_{i=1}^{\infty} A_i$ and so $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} B_i$. Careful inspection of this argument shows that it is also true that $\cup_{i=1}^{n} A_i = \cup_{i=1}^{n} B_i$ for each n. Finally,

$$\begin{aligned}
P\left(\cup_{i=1}^{\infty} A_i\right) &= P\left(\cup_{i=1}^{\infty} B_i\right) \\
&= \sum_{i=1}^{\infty} P(B_i) \quad \text{since the } B_n's \text{ are disjoint} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i) \quad \text{definition of infinite sum} \\
&= \lim_{n\to\infty} P\left(\cup_{i=1}^{n} B_i\right) \quad \text{since the} B_n's \text{ are disjoint} \\
&= \lim_{n\to\infty} P\left(\cup_{i=1}^{n} A_i\right) \quad \text{since } \cup_{i=1}^{n} A_i = \cup_{i=1}^{n} B_i \\
&= \lim_{n\to\infty} P(A_n) \quad \text{since } \cup_{i=1}^{n} A_i = A_n.
\end{aligned}$$

(2) We are given that $P(AB) = P(A)P(B)$. Now,

$$\begin{aligned}
P(A^c B^c) &= P((A \cup B)^c) \\
&= 1 - P(A \cup B) \\
&= 1 - P(A) - P(B) + P(AB) \\
&= 1 - P(A) - P(B) + P(A)P(B) \\
&= (1 - P(A))(1 - P(B)) \\
&= P(A^c)P(B^c).
\end{aligned}$$

(3a) $S = S_1 \cup S_2$ where

$$S_1 = \{s : \ s = (s_1, \ldots, s_m), \ s_1 = s_2 = \cdots = s_{m-1} = H \text{ and } s_m = T, \ m = 2, 3, \ldots, \}$$

$S_2 = \{s: \ s = (s_1, \ldots, s_m), \ s_1 = s_2 = \cdots = s_{m-1} = T \text{ and } s_m = H, \ m = 2, 3, \ldots, \}$.

(3b) Let $A =$ "*three tosses are required.*" Then $A = \{HHT, TTH\}$. So, $P(A) = P(\{HHT\}) + P(\{TTH\}) = P(H)P(H)P(T) + P(T)P(T)P(H) = (1/2)(1/2)(1/2) + (1/2)(1/2)(1/2) = 1/4$.

(4a) If $P(A) = 0$ then $P(AB) \leq P(A) = 0$. Hence, $0 = P(AB) = P(A)P(B)$ so A and B are independent. If $P(A) = 1$, then $1 \geq P(A \cup B) \geq P(A) = 1$ so $P(A \cup B) = 1$. Now $P(AB) = P(A) + P(B) - P(A \cup B) = 1 + P(B) - 1 = P(B) = 1 \times P(B) = P(A)P(B)$.

(4b) $P(A \cap A) = P(A)P(A)$. But $P(A \cap A) = P(A)$ so $P(A) = P(A)P(A)$. Thus, $P(A) = 0$ or 1.