Solutions to Homework 3

(1) Solution in lecture notes.

(2a) The brute force method is: list all 32 outcomes, and add up the relevant probabilities. A slightly more refined calculation is as follows. Let $A =$"at least one has blue eyes" and $B =$"at least three have blue eyes". Let $C_i =$"exactly i have blue eyes". Then $P(C_3) = P(\text{blue blue blue not blue not blue}) +$ $P(\text{blue blue not blue blue not blue}) + \cdots = 10(1/4)^3(3/4)^2$. Similarly, $P(C_4) = 5(1/4)^4(3/4)$ and $P(C_5) = (1/4)^5$. Also, $P(C_0) = (3/4)^5$.

Then

$$\begin{aligned} P(B|A) &= \frac{P(AB)}{P(A)} \\ &= \frac{P(B)}{P(A)} \\ &= \frac{P(C_3) + P(C_4) + P(C_5)}{1 - P(C_0)} \\ &= \frac{10(1/4)^3(3/4)^2 + 5(1/4)^4(3/4) + (1/4)^5}{1 - (3/4)^5} \\ &= .1357. \end{aligned}$$

(2b) At least 3 have blue eyes if and only if at least 2 of the other 4 have blue eyes. So we add up the probabilities of these sample outcomes:

$$6(1/4)^2(3/4)^2 + 4(1/4)^3(3/4) + (1/4)^4 = .2617.$$

(3) Since the A_i's form a partition, we know that $\sum_{i=1}^k P(A_i) = 1$ and $\sum_{i=1}^k P(A_i|B) = 1$. So,

$$\begin{aligned} 1 &= \sum_{i=1}^k P(A_i|B) \\ &< P(A_1) + \sum_{i=2}^k P(A_i|B) \end{aligned}$$

$$
\begin{aligned}
&= P(A_1) + \sum_{i=2}^{k}(P(A_i) + P(A_i|B) - P(A_i)) \\
&= \sum_{i=1}^{k} P(A_i) + \sum_{i=2}^{k}(P(A_i|B) - P(A_i)) \\
&= 1 + \sum_{i=2}^{k}(P(A_i|B) - P(A_i)).
\end{aligned}
$$

Therefore, $\sum_{i=2}^{k}(P(A_i|B) - P(A_i)) > 0$ so at least one of $P(A_i|B) - P(A_i) > 0$ for $i = 2, \ldots, k$.

(4) Use Bayes' theorem. Let C = conservative, L = liberal, I = independent and V = voted. Then,

$$
\begin{aligned}
P(L|V^c) &= \frac{P(V^c|L)P(L)}{P(V^c|C)P(C) + P(V^c|L)P(L) + P(V^c|I)P(I)} \\
&= \frac{(.18 \times .50)}{(.35 \times .30) + (.18 \times .50) + (.50 \times .20)} \\
&= .305.
\end{aligned}
$$

(5a)

$$
P(C_i|H) = \frac{P(H|C_i)P(C_i)}{\sum_{j=1}^{5} P(H|C_j)P(C_j)}
$$

Now, $\sum_{j=1}^{5} P(H|C_j)P(C_j) = (0 \times (1/5)) + ((1/4) \times (1/5)) + ((1/2) \times (1/5)) + ((3/4) \times (1/5)) + (1 \times (1/5)) = 1/2$. So,

$$
\begin{aligned}
P(C_1|H) &= \frac{0 \times (1/5)}{1/2} = 0 \\
P(C_2|H) &= \frac{(1/4) \times (1/5)}{1/2} = \frac{1}{10} \\
P(C_3|H) &= \frac{(1/2) \times (1/5)}{1/2} = \frac{2}{10} \\
P(C_4|H) &= \frac{0 \times (1/5)}{1/2} = \frac{3}{10}
\end{aligned}
$$

$$P(C_5|H) = \frac{1 \times (1/5)}{1/2} = \frac{4}{10}.$$

(5b) There are two ways to proceed. We could compute $P(C_i|A)$ by Bayes' theorem where $A = HH$ (two heads). Or we could repeat the calculation in (5a) using the posterior probabilities as prior probabilities, i.e. $p_1 = 0, p_2 = 1/10, p_3 = 2/10, p_4 = 3/10, p_5 = 4/10$. Now, $\sum_j (1/5)p_j = (0 \times (1/5)) + ((1/10) \times (1/5)) + ((2/10) \times (1/5)) + ((3/10) \times (1/5)) + ((4/10) \times (1/5)) = 1/5$. So,

$$\begin{aligned}
P(C_1|H) &= \frac{0 \times (1/5)}{1/5} = 0 \\
P(C_2|H) &= \frac{(1/10) \times (1/5)}{1/5} = \frac{1}{30} \\
P(C_3|H) &= \frac{(2/10) \times (1/5)}{1/5} = \frac{4}{30} \\
P(C_4|H) &= \frac{(3/10) \times (1/5)}{1/5} = \frac{9}{30} \\
P(C_5|H) &= \frac{(4/10) \times (1/5)}{1/5} = \frac{16}{30}.
\end{aligned}$$

(5c)

$$P(C_i|B_4) = \frac{P(B_4|C_i)P(C_i)}{\sum_{j=1}^{5} P(B_4|C_j)P(C_j)}.$$

Now, $P(B_4|C_1) = 0$, $P(B_4|C_2) = (3/4)^3(1/4) = 27/256$, $P(B_4|C_3) = (1/2)^3(1/2) = 1/8$, $P(B_4|C_4) = (1/4)^3(3/4) = 3/256$, $P(B_4|C_5) = 0$. So, $\sum_{j=1}^{5} P(B_4|C_j)P(C_j) = (0 \times (1/5)) + ((27/256) \times (1/5)) + ((1/8) \times (1/5)) + ((3/256) \times (1/5)) + (0 \times (1/5)) = .048$. Therefore,

$$\begin{aligned}
P(C_1|B_4) &= \frac{0 \times (1/5)}{.48} = 0 \\
P(C_2|B_4) &= \frac{(27/256) \times (1/5)}{.48} = 0.44 \\
P(C_3|B_4) &= \frac{(1/8) \times (1/5)}{.48} = 0.52
\end{aligned}$$

$$
\begin{aligned}
P(C_4|B_4) &= \frac{(3/256) \times (1/5)}{.48} = 0.05 \\
P(C_5|B_4) &= \frac{0 \times (1/5)}{.48} = 0
\end{aligned}
$$