Homework 4 Solutions

(1a) $P(2 < X \leq 4.8) = F(4.8) - F(2) = .2 - .1 = .1$. $P(2 \leq X \leq 4.8) = F(4.8) - F(2^-) = .2 - 0 = .2$.

(1b) $P(X = x) = P(X \leq x) - P(X < x) = F(x^+) - F(x^-)$.

(2) Let G be the cdf for X^+. Since $X^+ \geq 0$, $G(y) = 0$ for all $y < 0$. For $y \geq 0$, note that $\{X^+ \leq y\} = \{X \leq y\}$. Hence, $G(y) = P(X^+ \leq y) = P(X \leq y) = F(y)$. Therefore,

$$G(y) = \begin{cases} 0 & y < 0 \\ F(y) & y \geq 0. \end{cases}$$

(3a)

$$\begin{aligned}
\lim_{h\to 0} \frac{P(X \leq x+h | X > x)}{h} &= \lim_{h\to 0} \frac{P(X \leq x+h, X > x)}{h\,P(X > x)} \\
&= \lim_{h\to 0} \frac{P(x < X < x+h)}{h\,P(X > x)} \\
&= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \frac{1}{(1 - F(x))} \\
&= \frac{F'(x)}{1 - F(x)} \\
&= \frac{f(x)}{1 - F(x)}.
\end{aligned}$$

Interpretation: $h(x)dx$ is the probability of dying in $[x, x+dx]$ given that you survived up to x.

(3b) Since $h(u) = H'(u)$,

$$\begin{aligned}
\exp\left\{-\int_x^{x+y} h(u)\,du\right\} &= \exp\left\{-\int_x^{x+y} H'(u)\,du\right\} \\
&= \exp\{H(x) - H(x+y)\}
\end{aligned}$$

$$
\begin{aligned}
&= \exp\{\log(1-F(x+y)) - \log(1-F(x))\} \\
&= \frac{1-F(x+y)}{1-F(x)} \\
&= \frac{P(X > x+y)}{P(X > x)} \\
&= \frac{P(X > x+y, X > x)}{P(X > x)} \\
&= P(X > x+y | X > x).
\end{aligned}
$$

(3c) Since $X \sim \text{Exp}(\beta)$, $f(x) = \beta^{-1}e^{-x/\beta}$ for $x > 0$. Therefore, $F(x) = \int_0^x f(s)ds = 1-e^{-x/\beta}$. Hence, $h(x) = f(x)/(1-F(x)) = 1/\beta$. Interpretation: the probability of death in $[x, x+dx]$ you have lived up to x is the same, no matter what x is. This is a "memoryless process."

(4a) The density of X is

$$
f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}
$$

for $x \in \mathcal{R}$. Let $Y = e^X$. First note that $Y \geq 0$ so $F_Y(y) = 0$ if $y < 0$. For $y \geq 0$,

$$
\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(e^X \leq y) \\
&= P(X \leq \log y) \\
&= F_X(\log y) \\
&= \int_{-\infty}^{\log y} f_X(x)\,dx.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
f_Y(y) &= F'(y) \\
&= f_X(\log y)\frac{d \log y}{dy} \\
&= f_X(\log y)\frac{1}{y}
\end{aligned}
$$

$$= \frac{1}{y\sqrt{2\pi}} e^{-(\log y)^2/2}.$$

(5)

$$\begin{aligned}
F_R(r) &= P(R \leq r) \\
&= P(\sqrt{X^2+Y^2} \leq r) \\
&= P(X^2+Y^2 \leq r^2) \\
&= \frac{\text{area of disc of radius } r}{\text{area of disc of radius } 1} \\
&= \frac{\pi r^2}{\pi} \\
&= r^2.
\end{aligned}$$

Thus, $f_R(r) = 2r$ for $0 \leq r \leq 1$.

(6) Since X and Y are independent,

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \frac{e^{-x}\lambda^x}{x!} \frac{e^{-y}\mu^y}{y!}.$$

And, from the hint,

$$P(X+Y=n) = \frac{e^{-(\lambda+\mu)}(\lambda+\mu)^n}{n!}.$$

So,

$$\begin{aligned}
P(X=x|X+Y=n) &= \frac{P(X=x, Y=n-x)}{P(X+Y=n)} \\
&= \frac{\frac{e^{-x}\lambda^x}{x!} \frac{e^{-(n-x)}\mu^{n-x}}{(n-x)!}}{\frac{e^{-(\lambda+\mu)}(\lambda+\mu)^n}{n!}} \\
&= \binom{n}{x} \left(\frac{\lambda}{\lambda+\mu}\right)^x \left(\frac{\mu}{\lambda+\mu}\right)^{n-x} \\
&= \binom{n}{x} \pi^x (1-\pi)^{n-x}.
\end{aligned}$$

(7) First,

$$f_Y(y) = \int_0^1 f_{X,Y}(x,y)\,dx = c\int_0^1 (x+y^2)\,dx = c\left(\frac{1}{2}+y^2\right).$$

Therefore,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{c(x+y^2)}{c\left(\frac{1}{2}+y^2\right)} = \frac{x+y^2}{\frac{1}{2}+y^2}.$$

In particular,

$$f_{X,Y}\left(x \mid \frac{1}{2}\right) = \frac{x+\frac{1}{4}}{\frac{1}{2}+\frac{1}{4}} = \frac{4}{3}\left(x+\frac{1}{4}\right).$$

Finally,

$$P\left(X<\frac{1}{2} \mid Y=\frac{1}{2}\right) = \int_0^{1/2} f_{X,Y}\left(x \mid \frac{1}{2}\right)dx = \int_0^{1/2}\frac{4}{3}\left(x+\frac{1}{4}\right)dx = \frac{1}{2}.$$

(8a) .84

(8b) .89

(8c) 9.58

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(8a) pnorm(7,3,4) =  0.8413447

(8b)  1-pnorm(-2,3,4) = 0.8943502

(8c) qnorm(.95,3,4) =  9.579415

(8d) pnorm(4,3,4)-pnorm(0,3,4) = .372079

(8e) qnorm(.975,3,4) = 10.83986

(8f) ppois(12,10)-ppois(8,10) = 0.4587368
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or

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dpois(9,10) + dpois(10,10) + dpois(11,10) + dpois(12,10) =  0.4587368
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