Homework 5 Solutions

(1) Let F_n denote the cdf of Y_n. Then,

$$\begin{aligned}
F_n(y) &= P(Y_n \le y) \\
&= P(\max\{X_1, \ldots, X_n\} \le y) \\
&= P(X_1 \le y,\, X_2 \le y \ldots, X_n \le y) \\
&= \prod_{i=1}^{n} P(X_i \le y) \\
&= \prod_{i=1}^{n} F_X(y) \\
&= F_X(y)^n \\
&= y^n.
\end{aligned}$$

So, $f_Y(y) = ny^{n-1}$. Hence, $E(Y) = \int_0^1 y f_Y(y)\,dy = n\int_0^1 y\, y^{n-1}\,dy = n/(n+1)$.

(2) Let $Y_i = 1$ if the particle jumps to the right and $Y_i = -1$ if the particle jumps to the left. Thus, $P(Y_i = 1) = 1 - p$, $P(Y_i = -1) = p$ and $X_n = \sum_{i=1}^n Y_i$. So, $E(Y_i) = (1-p)(1) + (p)(-1) = 1 - 2p$ and $E(X_n) = \sum_i E(Y_i) = n(1-2p)$. Now, $E(Y_i^2) = (1-p)(1^2) + (p)((-1)^2) = 1$ and so $Var(Y_i) = E(Y_i^2) - (1-2p)^2 = 4p(1-p)$ and $V(X_n) = \sum_i V(Y_i) = 4np(1-p)$.

(3) $f_X(x) = 2^{-x}$ for $x = 1, 2, \ldots$. We want $E(X) = \sum_{x=1}^\infty x f_X(x) = \sum_{x=1}^\infty x(1/2)^x$. To evaluate this sum we use the following trick. For $0 < p1$ define $S(p) = \sum_{i=1}^\infty p^x$. Then $S'(p) = \sum_{i=1}^\infty x p^{x-1}$. In particular, $S'(1/2) = \sum_{i=1}^\infty x(1/2)^{x-1}$. But since $S(p) = \sum_{i=1}^\infty p^x = 1/(1-p)$, we see that $S'(p) = 1/(1-p)^2$ and so $S'(1/2) = 4$. So we have shown that $\sum_{i=1}^\infty x(1/2)^{x-1} = 4$. Now, $E(X) = \sum_{x=1}^\infty x(1/2)^x = \sum_{x=0}^\infty x(1/2)^x = (1/2)\sum_{x=0}^\infty x(1/2)^{x-1} = (1/2)4 = 2$.

(4) Let $A_y = \{x :\ r(x) = y\}$. Since $r(x) = y$ for all $x \in A_y$,

$$\sum_x r(x) f_X(x) = \sum_y \sum_{x \in A_y} r(x) f_X(x)$$

$$
\begin{aligned}
&= \sum_y \sum_{x \in A_y} y f_X(x) \\
&= \sum_y y \sum_{x \in A_y} f_X(x) \\
&= \sum_y y P(Y = y) = \sum_y f_Y(y) = E(Y).
\end{aligned}
$$

(5) $\int_0^\infty P(X > x)dx = \int_0^\infty [1 - F(x)]dx = \int u\, dv$ where $u = 1 - F(x)$ and $dv = 1$. So, $du = -f(x)$, $v = x$ and, integrating by parts,

$$
\begin{aligned}
\int_0^\infty P(X > x)dx &= \int_0^\infty [1 - F(x)]dx \\
&= \int u\, dv \\
&= [uv]_0^\infty - \int v\, du \\
&= [x(1 - F(x))]_0^\infty + \int x f(x) dx \\
&= \int x f(x) dx \\
&= E(X).
\end{aligned}
$$

Here is an alternative proof. Note that $x = \int_0^x dy$. Thus, $\int_0^\infty x f(x) dx = \int_0^\infty \int_0^x dy\, f(x) dx = \int_0^\infty \int_y^\infty dy f(x) dx = \int_0^\infty \int_y^\infty f(x) dx dy = \int_0^\infty (1 - F(y)) dy$.

(6) $\overline{X}_n$ is erratic for the Cauchy since $E(X)$ is not defined.

(7) Use the lazy statistician rule:

$$
\begin{aligned}
E(Y) &= \int_{-\infty}^\infty e^x \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \\
&= e^{1/2} \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2}\, dx \\
&= e^{1/2}
\end{aligned}
$$

since $\frac{1}{\sqrt{2\pi}}e^{-(x-1)^2/2}$ is the density for a $N(1,1)$ and therefore integrates to 1. Now,

$$
\begin{aligned}
E(Y^2) &= \int_{-\infty}^{\infty} e^{2x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \\
&= e^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-2)^2/2}\, dx \\
&= e^2.
\end{aligned}
$$

Thus, $V(Y) = E(Y^2) - (E(Y))^2 = e^2 - e = e(e-1)$.

(8a) From question (2), with $p = 1/2$, $E(X_n) = n(1-2p) = 0$ and $V(X_n) = 4np(1-p) = n$.

(8b) X_n has mean 0 but has huge variance. As a result, there can be wild variation from simulation to simulation.