Homework 6 Solutions

(1) Let Y represent the coin toss. So $P(Y=0) = P(Y=1) = 1/2$, $X|Y=0 \sim \text{Uniform}(3,4)$ and $X|Y=1 \sim \text{Uniform}(0,1)$.

(1a) $E(X|Y=0) = 7/2$, $E(X|Y=1) = 1/2$ and $E(Y) = 1/2$. So $E(X|Y) = (Y/2) + (1-Y)(7/2)$, and $E(X) = EE(X|Y) = E[(Y/2) + (1-Y)(7/2)] = E(Y)/2 + (1-E(Y))(7/2) = (1/4) + (7/4) = 2$.

(1b) $V(X|Y=0) = 1/12$, and $V(X|Y=1) = 1/12$. So $V(X|Y) = 1/12$. Therefore, $V(X) = VE(X|Y) + EV(X|Y) = V[(Y/2) + (1-Y)(7/2)] + E(1/12) = V((7/2) - 3Y) + (1/12) = 9V(Y) + (1/12) = (9/4) + (1/12) = 7/3$.

(2)

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\begin{aligned}
Cov\left(\sum_{i=1}^{m} a_i X_i,\ \sum_{j=1}^{n} b_j Y_j\right) &= E\left(\sum_{i=1}^{m} a_i X_i \ \sum_{j=1}^{n} b_j Y_j\right) - E\left(\sum_{i=1}^{m} a_i X_i\right) E\left(\sum_{j=1}^{n} b_j Y_j\right) \\
&= E\left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_i X_i b_j Y_j\right) - E\left(\sum_{i=1}^{m} a_i X_i\right) E\left(\sum_{j=1}^{n} b_j Y_j\right) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j E(X_i Y_j) - \sum_{i=1}^{m} a_i E(X_i) \sum_{j=1}^{n} b_j E(Y_j) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j E(X_i Y_j) - \sum_{i=1}^{m}\sum_{j=1}^{n} a_i E(X_i) b_j E(Y_j) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j \left[E(X_i Y_j) - E(X_i)E(Y_j)\right] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j \text{Cov}(X_i, Y_j).
\end{aligned}
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(3) $Var(2X - 3Y + 8) = 4Var(X) + 9Var(Y) - 12Cov(X,Y)$. Now, $f_X(x) = \int_0^2 (1/3)(x+y)dy = (2/3)(x+1)$ and $f_Y(y) = \int_0^1 (1/3)(x+y)dx = (1/3)(y + (1/2))$. Thus, $\mu_X = \int_0^1 x f_X(x)dx = 5/9$, $\sigma_X^2 = \int_0^1 x^2 f_X(x)dx - (5/9)^2 = (7/18) - (5/9)^2 = 13/162$ and $\mu_Y = \int_0^2 y f_Y(y)dy = 11/9$, $\sigma_Y^2 = \int_0^2 y^2 f_Y(y)dy - (11/9)^2 = 16/9 - (11/9)^2 = 23/81$. Also, $E(XY) = (1/3)\int_0^2\int_0^1 xy(x+y)dxdy = (1/3)\int_0^2\int_0^1 x^2 y dxdy + (1/3)\int_0^2\int_0^1 xy^2 dxdy = (1/3)\int_0^2 \left[\int_0^1 x^2 dx\right] ydy +$

$(1/3)\int_0^2\left[\int_0^1 xdx\right]y^2dy = (1/3)\int_0^2(1/3)ydy + (1/3)\int_0^2(1/2)y^2dy = 2/3$. So, $Cov(X,Y) = (2/3) - (5/9)(11/9) = -1/81$. Finally, $Var(2X-3Y+8) = 4Var(X)+9Var(Y)-12Cov(X,Y) = 4(13/162)+9(23/81)-12(-1/81) = 245/81$.

(4) $E(r(X)s(Y)|X=x) = \int r(x)s(y)f_{Y|X}(y|x)dy = r(x)\int s(y)f_{Y|X}(y|x)dy = r(x)E(s(Y)|X=x)$. Hence, $E(r(X)s(Y)|X) = r(X)E(s(Y)|X)$. For the second part, $E(r(X)|X=x) = \int r(x)f(y|x)dy = r(x)\int f(y|x)dy = r(x)$. So $E(r(X)|X) = r(X)$.

(5) As in the hint, let $m = E(Y)$ and let $b(X) = E(Y|X)$. So,

$$\begin{aligned} Var(Y) &= E(Y-m)^2 \\ &= E[(Y-b(X)) + (b(X)-m)]^2 \\ &= E(Y-b(X))^2 + E(b(X)-m)^2 + 2E[(Y-b(X))(b(X)-m)]. \end{aligned}$$

Now we evaluate each of these terms.

$$E(Y-b(X))^2 = EE[(Y-b(X))^2|X] = EE[(Y-E(Y|X))^2|X] = EVar(Y|X).$$

Before evaluating the second term, note that $b(X) = E(Y|X)$ so $m = E(Y) = EE(Y|X) = Eb(X)$, that is, m is the mean of the random variable $b(X)$. So $E(b(X)-m)^2 = \mathrm{Var}\, b(X) = \mathrm{Var} E(Y|X)$. Next,

$$\begin{aligned} E[(Y-b(X))(b(X)-m)] &= EE[(Y-b(X))(b(X)-m)|X] \\ &= E(b(X)-m)E[(Y-b(X))|X] \quad \text{from problem 4} \\ &= E(b(X)-m)\{E[Y|X]-b(X)\} \\ &= E(b(X)-m)\{b(X)-b(X)\} \\ &= 0. \end{aligned}$$

(6) Since $E(X|Y) = c$, $E(X) = EE(X|Y) = E(c) = c$. Also, $E(XY) = EE(XY|Y) = E[YE(X|Y)] = E(cY) = cE(Y)$. Hence, $Cov(X,Y) = E(XY) - E(X)E(Y) = cE(Y) - cE(Y) = 0$.

(7a) $E(\overline{X}_n) = E(n^{-1}\sum_i X_i) = n^{-1}\sum_i E(X_i) = n^{-1}\sum_i \mu = \mu$. $Var(\overline{X}_n) = n^{-2}\sum_i Var(X_i) = n^{-2}\sum_i \sigma^2 = \sigma^2/n$.

(7b) $E(\overline{X}_n) = 1/2$ and $Var(\overline{X}_n) = \sigma^2/n = (1/12n)$. The mean stays the same but the variance decreases in n. You might also notice that the distribution of $\overline{X}_n$ gets more concentrated around the mean and starts to look Normal.