Solutions for Homework 8

(1) Let $Y = n\hat{F}_n(x)$. Then $Y \sim \text{Binomial}(n, F(x))$. So, $\text{E}(Y) = nF(x)$, $\text{Var}(Y) = nF(x)(1 - F(x)$ and hence $\text{E}(\hat{F}_n(x)) = F(x)$ and $\text{Var}(\hat{F}_n(x)) = F(x)(1 - F(x)/n$. Note that $bias = 0$ and $\text{Var}(\hat{F}_n(x)) \to 0$ so $\text{E}(\hat{F}_n(x) - F(x)) \to 0$. This implies that $\hat{F}_n(x) \xrightarrow{p} F(x)$.

(2) Set $\epsilon_n^2 = \log(2/\alpha)/(2n)$ where $\alpha = .05$. From the DKW inequality, a 95 per cent interval is $\hat{F}_n(x) \pm \epsilon_n$.

(3) Without loss of generality assume that $x < y$. Let

$$I_i = \begin{cases} 1 & \text{if } X_i \leq x \\ 0 & \text{otherwise.} \end{cases}$$

and

$$J_i = \begin{cases} 1 & \text{if } X_i \leq y \\ 0 & \text{otherwise.} \end{cases}$$

So $\hat{F}_n(x) = n^{-1}\sum_{i=1}^n I_i$ and $\hat{F}_n(y) = n^{-1}\sum_{i=1}^n J_i$. Now, $\text{Cov}(\hat{F}_n(x), \hat{F}_n(y)) = \text{E}(\hat{F}_n(x)\hat{F}_n(y)) - \text{E}(\hat{F}_n(x))\text{E}(\hat{F}_n(y)) = \text{E}(\hat{F}_n(x)\hat{F}_n(y)) - F(x)F(y)$. And,

$$\begin{aligned} \text{E}(\hat{F}_n(x)\hat{F}_n(y)) &= n^{-2}\text{E}\left(\sum_i I_i\right)\left(\sum_i J_i\right) \\ &= n^{-2}\text{E}\left(\sum_i I_i J_i\right) + n^{-2}\text{E}\left(\sum_{i \neq j} I_i J_j\right) \\ &= n^{-2}\text{E}\left(\sum_i I_i\right) + n^{-2}\text{E}\left(\sum_{i \neq j} I_i J_j\right) \\ &= n^{-1}F(x) + n^{-2}\left(\sum_{i \neq j} \text{E}(I_i)\text{E}(J_j)\right) \\ &= n^{-1}F(x) + n^{-2}\sum_{i \neq j} F(x)F(y) \\ &= n^{-1}F(x) + n^{-2}n(n-1)F(x)F(y). \end{aligned}$$

Therefore,

$$\text{Cov}(\hat{F}_n(x), \hat{F}_n(y)) = n^{-1}F(x) + n^{-2}n(n-1)F(x)F(y) - F(x)F(y) = \frac{1}{n}\left[F(x) - F(x)F(y)\right].$$