Homework 9 Solutions

(1) $\overline{X} = 71$ and $se = S/\sqrt{n} = .82$. For the confidence interval, $z_{.05} = 1.645$ and $\overline{X} \pm 1.645\,\mathrm{se} = (70, 72)$. For the median, $\hat{\theta} = 76$. Everyone will get a different bootstrap standard error. I get $\hat{se} = .95$. The normal interval is $\hat{\theta} \pm 1.645\,\mathrm{se} = (74, 78)$. For the percentile interval I get $(74, 77)$.

(2) $\hat{p}_1 = 90/100$ and $\hat{p}_2 = 85/100$ so $\hat{\theta} = .90 - .85 = .05$. To compute the standard error:

$$\begin{aligned} \mathrm{Var}(\hat{p}_1 - \hat{p}_2) &= \mathrm{Var}(\hat{p}_1) + \mathrm{Var}(\hat{p}_2) \\ &= \frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}. \end{aligned}$$

Hence,

$$\mathrm{se} = \sqrt{\mathrm{Var}(\hat{p}_1 - \hat{p}_2)} = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

and the estimated standard error is

$$\hat{\mathrm{se}} = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}.$$

In this case we get

$$\hat{\mathrm{se}} = \sqrt{\frac{.9 \times .1}{100} + \frac{.85 \times .05}{100}} = .047.$$

An 80 per cent confidence interval is $.05 \pm 1.28\,\hat{se} = (-.01, .11)$. A 95 per cent confidence interval is $.05 \pm 2\,\hat{se} = (-.04, .14)$.

(3) $\hat{\theta} = 221.6 - 44.2 = 177.40$. From the bootstrap I get $\hat{se} = 64$ and a 95 per cent confidence interval $(37.45, 261.72)$.

(4) We may represent the bootstrap distribution as:

x^*	$P(X^* = x^*)$
X_1	$\frac{1}{n}$
X_2	$\frac{1}{n}$
$\vdots$	$\vdots$
X_n	$\frac{1}{n}$

Treat the X_i's as fixed. Now $E(X_i^*|X_1,\ldots,X_n) = \overline{X}$ so

$$E(\overline{X}^*|X_1,\ldots,X_n) = \frac{1}{n}\sum_i E(X_i^*) = \overline{X}$$

and

$$E(\overline{X}^*) = EE(\overline{X}^*|X_1,\ldots,X_n) = E(\overline{X}) = \mu$$

where $\mu = E(X_1)$. Now

$$\mathrm{Var}(\overline{X}^*|X_1,\ldots,X_n) = \frac{\mathrm{Var}(X_i^*)}{n}$$

and

$$\mathrm{Var}(X_i^*|X_1,\ldots,X_n) = \frac{\sum_i (X_i - \overline{X})^2}{n} = \frac{n-1}{n}S^2$$

where $S^2 = \sum_i (X_i - \overline{X})^2/(n-1)$. Hence,

$$\mathrm{Var}(\overline{X}^*|X_1,\ldots,X_n) = \frac{n-1}{n^2}S^2.$$

Also,

$$\begin{aligned}
\mathrm{Var}(\overline{X}^*) &= E\mathrm{Var}(\overline{X}^*|X_1,\ldots,X_n) + \mathrm{Var}E(\overline{X}^*|X_1,\ldots,X_n) \\
&= E\frac{n-1}{n^2}S^2 + \mathrm{Var}(\overline{X}) \\
&= \frac{n-1}{n^2}E(S^2) + \frac{\sigma^2}{n} \\
&= \frac{n-1}{n^2}\sigma^2 + \frac{\sigma^2}{n} \\
&= \frac{\sigma^2}{n}\left[2 - \frac{1}{n}\right] \\
&= \mathrm{Var}(\overline{X})\left[2 - \frac{1}{n}\right] \\
&\approx 2\mathrm{Var}(\overline{X}).
\end{aligned}$$