Solutions to Practice Test 1

(1) (i) $Q(A) = P(A|C) = P(AC)/P(C) \geq 0$. (ii) $Q(S) = P(S|C) = P(SC)/P(C) = P(C)/P(C) = 1$. (iii) Let $A_1, A_2, \ldots,$ be disjoint. Then $Q(\cup_i A_i) = P(\cup_i A_i|C) = P((\cup_i A_i) \cap C)/P(C) = P(\cup_i (A_i \cap C))/P(C) = \sum_i P(A_i \cap C)/P(C) = \sum_i P(A_i|C) = \sum_i Q(A_i)$.

(2) If $y < 0$ then $F_Y(y) = P(Y \leq y) = 0$. If $y \in [0,1)$, $F_Y(y) = P(Y \leq y) = P(Y = 0) = P(X < 1/2) = \int_0^{1/2} 2x dx = 1/4$. If $y \in [1,2]$, $F_Y(y) = P(Y \leq y) = P(0 < X < y/2) = \int_0^{y/2} 2x dx = y^2/4$. If $y > 2$ then $F_Y(y) = 1$. Hence,

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{4} & 0 \leq y < 1 \\ \frac{y^2}{4} & 1 < y \leq 2 \\ 1 & y > 2. \end{cases}$$

(3) Since $U = F_X(X) \sim$ Uniform$(0,1)$ and $Y = F_Y^{-1}(U) \sim F_Y$, it follows that $Y = F_Y^{-1}(F_X(X))$ has the desired distribution. Here are the details. Note that $F_X(x) = \int_0^x 2e^{-2t} dt = 1 - e^{-2x}$. Hence, $U = F_X(X) = 1 - e^{-2X} \sim$ Uniform$(0,1)$. Now use the inverse integral transform to create the random variable Y. Now, $F_Y(y) = \int_0^y e^{-t} dt = 1 - e^{-y}$ and $F_Y^{-1}(u) = -log(1-y)$. So $-log(1-U) \sim F_Y$. Putting this together, $-log(1-U) = -log(1-(1-e^{-2X})) = 2X \sim F_Y$. So $Y = 2X \sim F_Y$.

(4a) The sample space is

$$S = \{s = (s_1, s_2, \ldots, s_n) : \; s_i \in \{1, \ldots, 4\}, \; s_n \in \{5,6\}, \; n = 1, 2, \ldots\}.$$

(4b) Let $B_n =$"ends on toss n with a 5." Let $B =$"5 appears before 6." Thus, $B = \cup_{n=1}^{\infty} B_n$ and the B_n's are disjoint. Let $C_n =$"neither 5 or 6 on toss n" and let $D_n =$"5 on toss n." Note that $P(C_n) = P(\{1,2,3,4\}) = 2/3$ and $P(D_n) = 2/9$. So, $P(B_n) = P(C_1 C_2 \ldots C_{n-1} D_n) = P(C_1)P(C_2) \cdots P(C_{n-1})P(D_n) = (2/3)^{n-1}(1/3)$ and $P(B) = \sum_{n=1}^{\infty} P(B_n) = (2/9) \sum_{n=1}^{\infty} (2/3)^{n-1} = (2/9) \times 3 = 2/3$.

(5)

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ \frac{x}{2} & 0 < x \le 1 \\ \frac{1}{2} & 1 < x \le 3 \\ \frac{1}{2} + \frac{x-3}{2} & 3 < x \le 4 \\ 1 & x > 4. \end{cases}$$

(6) Let $A_z = [0, z] \times [0, z]$. Then, for $0 < z < 1$, $F_Z(z) = P(Z \le z) = P((X, Y) \in A_z) = z^2$. So, $f_Z(z) = 2z$ for $0 < z < 1$.

(7a) $\int_0^1 \int_0^1 c(x + y)dxdy = c((1/2) + (1/2)) = c$. Since f must integrate to 1, $c = 1$.

(7b) $f_X(x) = \int_0^1 f_{X,Y}(x, y)dy = \int_0^1 (x + y)dy = x + (1/2)$. Therefore, $f_{Y|X}(y|x) = f_{X,Y}(x, y)/f_X(x) = (x + y)/(x + (1/2))$.

(7c) $P(Y > (1/2)|X = 1) = \int_{1/2}^1 f(y|1)dy = \int_{1/2}^1 \frac{2}{3}(1 + y)dy = 7/12$.

(7d) $P(Y > (1/2)|X < (1/2)) = P(Y > (1/2),\ X < (1/2))/P(X < (1/2))$. Now, $P(Y > (1/2),\ X < (1/2)) = \int_{1/2}^1 \int_0^{1/2} (x + y)dxdy = 2/3$.

(8) First, note that $1 = \int \int f_{X,Y}(x, y)dxdy = \int \int g(x)h(y)dxdy = \int g(x)dx \int h(y)dy$. Now, $f_X(x) = \int f_{X,Y}(x, y)dy = \int g(x)h(y)dy = g(x) \int h(y)dy$ and $f_Y(y) = \int f_{X,Y}(x, y)dy = \int g(x)h(y)dx = h(y) \int g(x)dx$. Therefore, $f_X(x)f_Y(y) = g(x)h(y) \int h(y)dy \int g(x)dx = g(x)h(y) = f_{X,Y}(x, y)$. Therefore, X and Y are independent.

(9) X only takes values on $(0, 1)$ as does Y. For $0 < y < 1$, $F_Y(y) = P(Y \le y) = P(X^2 < y) = P(X < \sqrt{y}) = \int_0^{\sqrt{y}} f_X(x)dx = \sqrt{y}$. So, $f_Y(y) = F'(y) = (1/2)y^{-1/2}$.