Solutions to Practice Final Exam

(1) The cdf is

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & 0 \le x < 1 \\ \frac{1}{2} & 1 \le x < 3 \\ \frac{1}{2} + \frac{x-3}{2} & 3 \le x < 4 \\ 1 & x \ge 4. \end{cases}$$

(2) We know that

$$\rho(X_i, X_j) = \frac{\text{Cov}(X_i, X_j)}{\sqrt{\text{Var}(X_i)\text{Var}(X_j)}} = \frac{\text{Cov}(X_i, X_j)}{\sigma^2}$$

so that $\text{Cov}(X_i, X_j) = \sigma^2\rho$. Hence,

$$\begin{aligned} Var(\sum_i X_i) &= \sum_i Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j) \\ &= \sum_i Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j) \\ &= \sum_i Var(X_i) + 2\sum_{i<j} \sigma^2\rho \\ &= n\sigma^2 + n(n-1)\sigma^2\rho. \end{aligned}$$

Since, $Var(\sum_i X_i) \ge 0$, conclude that

$$n\sigma^2 + n(n-1)\sigma^2\rho \ge 0$$

and hence $\rho \ge -1/(n-1)$.

(3) Let $Z \sim N(0,1)$. Then,

$$\begin{aligned} \psi &= E(Y_1) = (1 \times P(X > 0)) + (-1 \times P(X < 0)) \\ &= P(X > 0) - P(X < 0) \\ &= (1 - P(X < 0)) - P(X < 0) \\ &= 1 - 2P(X < 0) \\ &= 1 - 2P\left(X - \theta < -\theta\right) \\ &= 1 - 2P\left(Z < -\theta\right) \\ &= 1 - 2\Phi(-\theta) \equiv g(\theta). \end{aligned}$$

(3a) The mle is $\widehat{\psi} = g(\widehat{\theta}) = g(\overline{X}) = 1 - 2\Phi(-\overline{X})$.

(3b) The estimated standard error of $\widehat{\theta}$ is $se(\widehat{\theta}) = 1/\sqrt{n}$. Now use the delta method: $g\prime(\theta) = 2\phi(-\theta) = 2\phi(\theta)$ and the estimated standard error of $\widehat{\psi}$ is $se(\widehat{\psi}) = se(\widehat{\theta})|g'(\widehat{\theta})| = 2\phi(\overline{X})/\sqrt{n}$. The approximate confidence interval is

$$1 - 2\Phi(-\overline{X}) \pm 2\left(2\phi(\overline{X})/\sqrt{n}\right).$$

(3c) The variance of $\widetilde{\psi}$ is

$$\text{Var}(\widetilde{\psi}) = \frac{Var(Y_1)}{n}.$$

Now, $Var(Y_1) = E(Y_1^1) - \psi^2 = 1 - \psi^2$. The ARE is

$$\frac{se(\widehat{\psi})}{se(\widetilde{\psi})} = \frac{2\phi(\theta)}{\sqrt{1-\psi^2}} = \frac{\phi(\theta)}{\sqrt{\Phi(\theta)(1-\Phi(\theta))}}.$$

(4) Note that $f(X_i;\theta)$ is 0 if $X_i > \theta$ or $X_i < -\theta$. The likelihood will be 0 if this happens for any X_i. So $\mathcal{L}(\theta)$ is non-zero only if $-\theta \leq X_i \leq \theta$ for all X_i. This is the same as $\max_i |X_i| \leq \theta$. On the other hand, $f(X_i;\theta) = 1/(2\theta)$ if $-\theta \leq X_i \leq \theta$. So,

$$\mathcal{L}_n(\theta) = \begin{cases} \left(\frac{1}{2\theta}\right)^n & \theta \geq \max_i |X_i| \\ 0 & \text{otherwise.} \end{cases}$$

(4b) $\widehat{\theta} = max_i|X_i|$.

(4c) First, $0 \leq \widehat{\theta} \leq \theta$. For $c \in [0,\theta]$,

$$H(c) = P(\widehat{\theta} \leq c)$$

$$
\begin{aligned}
&= P(\max_i |X_i| \le c) \\
&= \prod_i P(|X_i| \le c) \\
&= \prod_i P(-c \le X_i \le c) \\
&= \prod_i \int_{-c}^{c} f(x;\theta)dx \\
&= \prod_i \frac{c}{\theta} \\
&= \left(\frac{c}{\theta}\right)^n.
\end{aligned}
$$

The density is $h(c) = n\theta^{-n}c^{n-1}$ for $c \in [0,\theta]$.

(5a) $\mathcal{L}_n(p) = p^Y(1-p)^{n-Y}$ where $Y = \sum_i X_i$ and $\ell(p) = Y\log p + (n - Y)\log(1-p)$. Setting $\ell'(p) = 0$ yields $\widehat{p} = Y/n$.

(5b) Let $f(x;p) = p^x(1-p)^{1-x}$. The Fisher information is

$$I(p) = E\left(-\frac{\partial^2 \log f}{\partial p^2}\right) = \frac{1}{p(1-p)}.$$

(5c) $\widehat{p} \approx N(p, 1/(nI(p))) = N(p, p(1-p)/n)$.

(5d) $MSE = \text{bias}^2 + \text{var} = 0 \; + \; p(1-p)/n = p(1-p)/n$.

(6a) Let $Q = \sum_i (Y_i - \beta x_i)^2$. Setting $dQ/d\beta = 0$ gives $\widehat{\beta} = \sum_i x_i Y_i / \sum_i x_i^2$.

(6b) $E(\widehat{\beta}) = \sum_i x_i E(Y_i)/\sum_i x_i^2 = \beta \sum_i x_i x_i / \sum_i x_i^2 = \beta \sum_i x_i^2/\sum_i x_i^2 = \beta$ as long as $\sum_i x^2 \neq 0$ i.e. as long as $x_i \neq 0$ for some x_i.

(6c) The variance is $\text{Var}(\widehat{\beta}) = \sum_i x_i^2 Var(Y_i)/(\sum_i x_i^2)^2 = \sigma^2 \sum_i x_i^2/(\sum_i x_i^2)^2 = \sigma^2/(\sum_i x_i^2)$. If $(\sum_i x_i^2) \to \infty$ then $MSE \to 0$ and then $\widehat{\beta} \xrightarrow{p} \beta$.

(7a) First,

$$F_Y(0) = P(Y \leq 0) = P(Y = 0) = P(X \leq 1) = \int_0^1 e^{-x} dx = 1 - e^{-1}.$$

For $0 < y < 1$, $F_Y(y) = F_Y(0)$. For $y \geq 1$, $F_Y(y) = P(Y \leq y) = P(Y = 0) + P(1 \leq Y \leq y) = P(Y = 0) + P(1 \leq X \leq y) = 1 - e^1 + \int_1^y e^{-x} dx = 1 - e^1 + (1 - e^{-y}) = 2 - e^{-1} - e^{-y}$.

(7b) Write $Y = r(X)$ where $r(x) = 0$ for $0 \leq x \leq 1$ and $r(x) = x$ for $x \geq 1$. So,

$$E(Y) = \int_0^\infty r(x) e^{-x} dx = \int_1^\infty x e^{-x} dx = \frac{2}{e}.$$

(7c) Given $X = x$, $Y = r(X)$ is a point mass at $r(x)$. So $E(Y|X = x) = r(x)$. Hence, $E(Y|X) = r(X)$.

(8a) Let $X_{(1)}$ and $X_{(n)}$ be the smallest and largest values. The likelihood is 0 unless $\theta < X_{(1)}$ and $2\theta > X_{(n)}$ i.e. $X_{(n)}/2 \leq \theta \leq X_{(1)}$. Thus,

$$\mathcal{L}_n(\theta) = \begin{cases} \left(\frac{1}{\theta}\right)^n & \frac{X_{(n)}}{2} \leq \theta \leq X_{(1)} \\ 0 & \text{otherwise.} \end{cases}$$

This is a decreasing function so $\widehat{\theta} = X_{(1)}$.

(8b) First, $\theta \leq \widehat{\theta} \leq 2\theta$. For $c \in [\theta, 2\theta]$,

$$\begin{aligned} H(c) &= P(\widehat{\theta} \leq c) \\ &= P(\min_i X_i \leq c) \\ &= 1 - P(\min_i X_i \geq c) \\ &= 1 - \prod_i P(X_i \geq c) \end{aligned}$$

$$
\begin{aligned}
&= 1 - \prod_i \int_c^{2\theta} f(x;\theta)dx \\
&= 1 - \prod_i \frac{2\theta - c}{\theta} \\
&= 1 - \left(\frac{2\theta - c}{\theta}\right)^n.
\end{aligned}
$$

The density is $h(c) = (n/\theta)(2 - (c/\theta))^{n-1}$ for $c \in [\theta, 2\theta]$.

(8c) For $n = 2$ the density is $h(c) = (2/\theta)(2 - (c/\theta))$. The mean is

$$
\int_\theta^{2\theta} c\, h(c) dc = \frac{4\theta}{3}.
$$

(9) Parts (i)-(iii) should be straightforward. For part (iv), note that $Y = \sum_i X_i \sim \text{Poisson}(n\lambda)$. Let $h(c) = P(\widehat{\lambda} = c)$. Then, if nc is a nonnegative integer,

$$
h(c) = P(Y = nc) = \frac{e^{-n\lambda}(n\lambda)^{nc}}{(nc)!}.
$$

For part (v), note that $se(\widehat{\lambda}) = 1/\sqrt{nI(\widehat{\lambda})} = \sqrt{\widehat{\lambda}/n} = \sqrt{\overline{X}/n}$. Also, $\psi = e^{-\lambda} \equiv g(\lambda)$. So $se(\widehat{\psi}) = se(\widehat{\lambda})|g'| = \sqrt{\overline{X}/n}e^{-\overline{X}}$. The confidence interval is

$$
e^{-\overline{X}} \pm 2\sqrt{\overline{X}/n}e^{-\overline{X}}.
$$

(10) Let $Y = \sum_i X_i$. The posterior density is

$$
f(\lambda|X^n) \propto \frac{1}{\sqrt{\lambda}} e^{-n\lambda}\lambda^Y = e^{-n\lambda}\lambda^{Y-(1/2)}.
$$

This is Gamma $(Y + (1/2), 1/n)$. The mean of a Gamma (α, β) is (in this version) $\alpha\beta$. So the posterior mean is Y/n.