Solutions to Practice Test 2

(1) $X \sim \text{Binomial}(25, .9)$, so $X = \sum_{i=1}^{n} X_i$ where $X_i \sim \text{Bernoulli}(p)$, $p = .9$, $n = 25$. Now $\mu = E(X_1) = p = .9$ and $\sigma^2 = \text{Var}(X_1) = p(1-p) = .9(.1) = .09$ so $\sigma = .3$. Let $Z \sim N(0,1)$. Then,

$$\begin{aligned}
P(X > 24) &= P(\sum_i X_i > 24) \\
&= P\left(\overline{X}_n > \frac{24}{25}\right) \\
&= P\left(\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} > \frac{\sqrt{n}(\frac{24}{25} - \mu)}{\sigma}\right) \\
&= P\left(\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} > \frac{\sqrt{25}(\frac{24}{25} - .9)}{.3}\right) \\
&\approx P\left(Z > \frac{\sqrt{25}(\frac{24}{25} - .9)}{.3}\right) \\
&= P(Z > 1) = 1 - \Phi(1) = 0.16.
\end{aligned}$$

(2) In this case, $E(X_1) = \lambda = 1$ and $\text{Var}(X_1) = \lambda = 1$. So,

$$\begin{aligned}
P(Y < 90) &= P(\sum_i X_i < 90) \\
&= P\left(\overline{X}_n < .9\right) \\
&= P\left(\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} < \frac{\sqrt{n}(.9 - \mu)}{\sigma}\right) \\
&= P\left(\frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} < \frac{\sqrt{25}(.9 - 1)}{1}\right) \\
&\approx P\left(Z < -.1\right) \\
&= P\left(Z > .1\right) = 1 - P\left(Z < .1\right) = 1 - \Phi(1) = .16
\end{aligned}$$

(3) $X_n \xrightarrow{p} X$ if, for every $\epsilon > 0$, $P(|X_n - X| > \epsilon) \to 0$ as $n \to \infty$. $X_n \xrightarrow{d} X$ if, $F_n(x) \to F(x)$ as $n \to \infty$, at all x at which F is continuous.

$X_n \xrightarrow{p} X$ always implies that $X_n \xrightarrow{d} X$. For the reverse direction, we have that $X_n \xrightarrow{d} X$ implies $X_n \xrightarrow{p} X$ if $P(X = c) = 1$ from some c.

(4) Fix $\epsilon > 0$. Then $|X_n - X| > \epsilon$ only if $X = e^n$ which happens with probability $1/n$. So, $P(|X_n - X| > \epsilon) = 1/n \to 0$. Therefore, $X_n \xrightarrow{p} X$. Since convergence in probability implies convergence in distribution, we also have that $X_n \xrightarrow{d} X$. To see if $X_n \xrightarrow{q.m.} X$, note that $(X - X_n)^2 = 0$ when $X = X_n$ which occurs with probability $(1 - (1/n))$. When $X_n \neq X$, $(X - X_n)^2 = (e^n - 1)^2$ with probability $1/2$ and $(X - X_n)^2 = (e^n + 1)^2$ with probability $1/2$. So

$$E(X - X_n)^2 = \frac{1}{n}\left(\frac{1}{2}(e^n - 1)^2 + \frac{1}{2}(e^n + 1)^2\right).$$

Since $e^n/n \to \infty$, we see that $E(X - X_n)^2 \to \infty$ as $n \to \infty$. Thus, X_n does not converge in quadratic mean to X.

(5) Using Markov's inequality,

$$P(|Z| > t) = P(|Z|^k > t^k) \leq \frac{E|Z|^k}{t^k}.$$

(5b)

$$\begin{aligned}
P(|Z| > t) &= 2P(Z > t) \\
&= 2\int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&\leq 2\int_t^\infty \frac{x}{t}\frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&= \frac{2}{\sqrt{2\pi}t}\int_t^\infty x e^{-x^2/2} dx \\
&\qquad v = e^{-x^2/2}, \quad dv = -xe^{-x^2/2} \\
&= \frac{2}{\sqrt{2\pi}t}\int_0^{e^{-t^2/2}} dv \\
&= \sqrt{\frac{2}{\pi}}\frac{e^{-t^2/2}}{t}.
\end{aligned}$$

(6) First note that X is a point mass at 0, i.e. $P(X = 0) = 1$. Also, $\sqrt{n}X_n \sim N(0,1)$. Let $Z \sim N(0,1)$. Then,

$$\begin{aligned} P(|X_n| > \epsilon) &= P(\sqrt{n}|X_n| > \sqrt{n}\epsilon) \\ &= P(|Z| > \sqrt{n}\epsilon) \to 0 \end{aligned}$$

since $\sqrt{n}\epsilon \to \infty$. Hence, $X_n \xrightarrow{p} X$. Since convergence in probability implies convergence in distribution, we also have that $X_n \xrightarrow{d} X$.

(7) Suppose that $X_n \xrightarrow{d} X$. Let F_n denote the cdf of X_n and let F denote the cdf of X. Every non-integer x is a point of continuity of F. So, for every integer k, $F_n(k+\epsilon) \to F(k+\epsilon)$ for any $0 < \epsilon < 1$. Now,

$$\begin{aligned} P(X_n = k) &= F_n(k+\epsilon) - F_n(k-\epsilon) \\ &\to F(k+\epsilon) - F(k-\epsilon) \\ &= P(X = k). \end{aligned}$$

Now suppose that $P(X_n = k) \to P(X = k)$. Let x be a point of continuity of F. Then x is not an integer, so $x = k + \epsilon$ for some integer k and some $0 < \epsilon < 1$.

$$F_n(x) = P(X_n \le x) = \sum_{j=1}^{k} P(X_n = j) \to \sum_{j=1}^{k} P(X = j) = P(X \le k) = P(X \le x) = F(x).$$

Thus, $X_n \xrightarrow{d} X$.

(8) Let F_n be the cdf of X_n. Then,

$$\begin{aligned} P(X_n \le x) &= P(n \min\{Z_1, \ldots, Z_n\} \le x) \\ &= P\left(\min\{Z_1, \ldots, Z_n\} \le \frac{x}{n}\right) \\ &= 1 - P\left(\min\{Z_1, \ldots, Z_n\} > \frac{x}{n}\right) \\ &= 1 - P\left(Z_i > \frac{x}{n}, \text{ for all } i\right) \\ &= 1 - \prod_i P\left(Z_i > \frac{x}{n}\right) \end{aligned}$$

$$
\begin{aligned}
&= 1-\left[P\left(Z_1>\frac{x}{n}\right)\right]^n \\
&= 1-\left[1-P\left(Z_i<\frac{x}{n}\right)\right]^n \\
&= 1-\left[1-F\left(\frac{x}{n}\right)\right]^n \\
&= 1-\exp\left\{\frac{\log[1-F(x/n)]}{\frac{1}{n}}\right\}.
\end{aligned}
$$

By L'Hopital's rule, the second term converges to $e^{-xf(0)} = e^{-\lambda x}$. So, $F_n(x) \to 1 - e^{-\lambda x}$ which is the cdf of an exponential random variable with mean $1/\lambda$.

(9) For any fixed x, $p_n(x) = 0$ for all large n. Therefore, $p(x) = 0$ for all x so $p(x)$ is not a probability function. However, $P(|X_n| > \epsilon) = 0$ for all large n. Hence, $X_n \xrightarrow{p} 0$.