Solutions: Test 2

Let $Y_n = n^{-1}\sum_{i=1}^n X_i^2$. Recall that $Y_n \overset{q.m}{\to} p$ if and only if $E(Y_n) \to p$ and $Var(Y_n) \to 0$. Now $E(Y_n) = E(X_1^2) = (p \times 1^2) + ((1-p) \times 0^2) = p$. So we only need to show that $Var(Y_n) \to 0$. First note that $E(X_1^4) = (p \times 1^4) + ((1-p) \times 0^4) = p$. Now $Var(Y_n) = Var(Y_1)/n = [E(Y_1^2) - E(Y_1)^2]/n = [E(X_1^4) - p^2]/n = [p - p^2]/n \to 0$.

(2) By the CLT, $\overline{X} \approx N(68, 16/n)$. So $P(\overline{X} > 68) = P(\sqrt{n}(\overline{X} - 68)/\sigma > 0) \approx P(Z > 0) = 1/2$.

(3) $E(Y) = E(E(Y|X)) = E(X)$ and $E(XY) = EE(XY|X) = EXE(Y|X) = EX^2$ since $E(Y|X) = X$. Hence, $Cov(X,Y) = E(XY) - E(X)E(Y) = E(X^2) - E(X)^2 = Var(X)$.

(4a) $P(X_n > \epsilon) \leq E(X_n)/\epsilon = 1/(n\epsilon) \to 0$.

(4b) $P(Y_n = 0) = P(nX_n = 0) = P(X_n = 0) = e^{-\lambda_n} = e^{-1/n} \to 1$. Hence, $P(Y_n > \epsilon) \to 0$.