Practice Test 3
36-325/725

(1a) Construct the Wald test for testing

$$H_0 : p = p_0 \quad \text{versus} \quad H_1 : p \neq p_0$$

where $p_0 \in (0,1)$.

(b) Derive an approximate expression for the (asymptotic) power of this test.

(2) Let $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$ where both μ and σ are unknown. Construct the likelihood ratio test for testing

$$H_0 : \sigma = 1 \quad \text{versus} \quad H_1 : \sigma \neq 1.$$

(3) Let $X_1, \ldots, X_n$ be i.i.d. observations from a Poisson (λ) distribution. Consider the prior probability density function $f(\lambda) = 1/\sqrt{\lambda}$.

(a) Find the posterior distribution for λ.

(b) Let $\overline{\lambda} = E(\lambda | X_1, \ldots, X_n)$ be the Bayes estimator. Show that

$$\overline{\lambda} = \frac{\sum_i X_i + \frac{1}{2}}{n}.$$

(4) Let $X_1, \ldots, X_n$ be iid with density $f(x;\, \beta) = \beta e^{-\beta x}$ for $x > 0$ and $\beta > 0$.

(a) Find the asymptotic (large sample) likelihood ratio test (LRT) of size α for $H_0 : \beta = \beta_0$ versus $H_1 : \beta \neq \beta_0$.

(b) Find an exact test, that is, a test with size equal to α (no asymptotic approximation).

(c) Find the power of the test in part (b).

(5) Let $X_1, \ldots, X_n \sim F$ and let $\mu = \mathbb{E}_F(X_1)$. We want to test $H_0 : \mu = 0$ versus $H_1 : \mu \neq 0$.

(a) Assume the data are $N(\mu, \sigma^2)$ where σ^2 is known. Find the Wald test.
(b) Repeat (a) but now assume that σ^2 is not known.
(c) Explain how to use the nonparametric bootstrap to construct a Wald test without assuming Normality.
(d) Explain how to use the parametric bootstrap to construct a Wald test assuming Normality.

(6) Let θ be a scalar parameter and consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. Suppose we test this hypothesis by rejecting H_0 when $|U| > z_{\alpha/2}$ where

$$U = \frac{f(\widehat{\theta}) - f(\theta_0)}{\widehat{se}(\widehat{\theta})|f'(\widehat{\theta})|}$$

and f is an aribtrary, smooth, monotone function. Show that asymptotically this is a size α test.

(7) Consider the following data: $X = 2$ and $Y = (Y_1, Y_2) = (1, 5)$ and suppose we want to test the null hypothesis that these two samples are drawn from the same distribution. Let the test statistic be $T = |\overline{X} - \overline{Y}|$. Find the permutation distribution of T. Find the p-value.

(8) Let $X_1, \ldots, X_n \sim N(\theta, 1)$.

(a) Construct the likelihood ratio test (LRT) of size α for $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$.
(b) Show that the test in (a) is exact in the sense that the size is exactly α for all n.

(9) True or false.

(a) The p-value of a test is the type I error.

(b) The type II error of a test is one minus the probability of falsely rejecting the null hypothesis.

(c) The p-value of a test has a Uniform(0,1) distribution.

(d) The nonparametric bootstrap can be used to construct a likelihood ratio test.

(e) The posterior probability that H_1 is true is one minus the p-value.

(f) When conducting k hypothesis tests, each based on n observations, the Bonferroni adjustment replaces α with α^k.

(10) Let $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$.

(a) Find the Jeffreys prior for (μ, σ).

(b) Find the posterior for (μ, σ).

(c) Find the posterior for μ.

(11) Chapter 13 exercise 1.

(12) Chapter 13 exercise 2.

(13) Chapter 13 exercise 3.

(14) Chapter 13 exercise 4.

(15) Chapter 13 exercise 5.