Practice Test 1
36-325/725
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1. Let X have probability density function
$$f_X(x) = \begin{cases} 1/2 & 0 < x < 1 \\ 1/2 & 3 < x < 4 \\ 0 & \text{otherwise.} \end{cases}$$
Find the cumulative distribution function of X.

2. Let X have distribution F and density function f and let A be a subset of the real line. Let $I_A(x)$ be the indicator function for A:
$$I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$
Let $Y = I_A(X)$. Find an expression for the cumulative distribution of Y. (Hint: first find the probability mass function for Y.)

3. Let (X, Y) have a uniform distribution over the unit square. Let $Z = \log(X) + \log(Y)$. Find the density $f_Z(z)$ for Z.

4. Let Ω be a sample space and let $C \subset S$ be such that $\mathbb{P}(C) > 0$. Let $Q(A) = \mathbb{P}(A \mid C)$. Show that Q satisfies the axioms of probability.

5. Let X have probability density function
$$f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$
Let $Y = 0$ if $X < 1/2$ and $Y = 2X$ otherwise. Find the distribution function of Y.

6. Let X have probability density function
$$f_X(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$
Find a random variable $Y = r(X)$ such at Y has density
$$f_Y(y) = \begin{cases} e^{-y} & y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

7. Let X and Y have joint density

$$f_{X,Y}(x,y) = \begin{cases} c(x+y) & 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find c.
   (b) Find $f_{Y|X}(y|x)$.
   (c) Find $\mathbb{P}(Y > 1/2 | X = 1)$.
   (d) Find $\mathbb{P}(Y > 1/2 | X < 1/2)$.

8. Let X and Y be random variables. Suppose that $\mathbb{E}(Y|X) = X$. Show that $\mathsf{Cov}(X,Y) = \mathbb{V}(X)$.

9. Let $X \sim \text{uniform}(0,1)$. Let $0 < a < b < 1$. Let

$$Y = \begin{cases} 1 & 0 < x < b \\ 0 & \text{otherwise} \end{cases}$$

   and let

$$Z = \begin{cases} 1 & a < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Are Y and Z independent? Why/Why not?
   (b) Find $\mathbb{E}(Y|Z)$.