Practice Test 2
36-325/725
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1. Let $X_1, \ldots, X_n \sim \text{Bernoulli}(p)$. Prove that
$$\frac{1}{n}\sum_{i=1}^n X_i^8 \xrightarrow{\text{P}} p \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^n X_i^8 \xrightarrow{\text{qm}} p.$$

2. Suppose that the height of men has mean 68 inches and standard deviation 4 inches while the height of women has mean 64 inches and standard deviation 3 inches. We draw 100 men and 100 women at random from the population. Find (approximately) the probability that the average height of men in our sample will be larger than that the average height of women in our sample.

3. Let $\lambda_n = 1/n$ for $n = 1, 2, \ldots$. Let $X_n \sim \text{Poisson}(\lambda_n)$. Show that $X_n \xrightarrow{\text{P}} 0$. Let $Y_n = nX_n$. Show that $Y_n \xrightarrow{\text{P}} 0$.

4. Let $X_1, \ldots, X_n \sim \text{Bernoulli}(p)$.

   (a) Find the MLE for p and the Fisher information.

   (b) Use the delta method to find the standard error of $\widehat{\psi} = e^{\widehat{p}}$. Find an 86 per cent confidence interval for ψ.

   (c) Explain the steps for getting the standard error of ψ using the parametric bootstrap.

   (d) Show that the parametric and nonparametric bootstrap are equivalent in this model.

5. We have n plants. The probability that any plant will produce a flower is p. Of those that flower, a fraction q will produce fruit.

   (a) Find the likelihood function for $\theta = (p, q)$.

   (b) Find the MLE and the method of moments estimator.

   (c) Find the Fisher information matrix.

   (d) Find an 80 per cent confidence interval for $\psi = pq$.

6. Let $X_n \sim \text{Normal}(1/n, 1/n)$, $n = 1, 2, \ldots$. Show that $X_n \xrightarrow{\text{P}} 0$.

7. Let $X_n \sim \text{Normal}(1/n, n)$, $n = 1, 2, \ldots$. Show that there is no random variable X such that $X_n \xrightarrow{\text{P}} X$.

8. Suppose that $X_n \sim N(0, 1/n)$ and let X be a random variable with distribution $F(x) = 0$ if $x < 0$ and $F(x) = 1$ if $x \geq 0$. Does X_n converge to X in probability? (Prove or disprove). Does X_n converge to X in distribution? (Prove or disprove).

9. Let $X, X_1, X_2, X_3, \ldots$ be random variables that are positive and integer valued. Show that $X_n \xrightarrow{d} X$ if and only if

$$\lim_{n\to\infty} \mathbb{P}(X_n = k) = \mathbb{P}(X = k)$$

for every integer k.

10. Let X_n be a random variable with probability mass function

$$p_n(x) = \begin{cases} \frac{1}{2} & \text{if } x = -\left(\frac{1}{2}\right)^n \\ \frac{1}{2} & \text{if } x = \left(\frac{1}{2}\right)^n \\ 0 & \text{otherwise.} \end{cases}$$

Let $p(x) = \lim_{n\to\infty} p_n(x)$. Is $p(x)$ a probability function? Does X_n converge in distribution to some random variable?

11. Let

$$\psi = \frac{\mathbb{E}(X^5)\mathbb{E}(X^4)}{\mathbb{E}(X^3)}.$$

Find the plug-in estimator of ψ. Explain how to get the standard error using the nonparametric bootstrap.

12. Let $X_1, \ldots, X_n \sim \text{Bernoulli}(p)$. Write an expression expression for the empirical cdf $\widehat{F}_n$. Show that

$$\max_x |\widehat{F}_n(x) - F(x)| \xrightarrow{\text{P}} 0.$$

13. Let $X \sim \text{Multinomial}(n, p)$ where $X = (X_1, \ldots, X_k)$ and $p = (p_1, \ldots, p_k)$. Find the mle and the Fisher information matrix.