Solutions: Test 3

(1a) The mle's are $\hat{\lambda} = \overline{X}$ and $\hat{\nu} = \overline{Y}$. The mle of $\delta = \lambda - \nu$ is $\hat{\delta} = \overline{X} - \overline{Y}$. The estimated standard error of $\hat{\delta}$ is

$$\hat{se} = \sqrt{\frac{\overline{X}}{n} + \frac{\overline{Y}}{m}}.$$

The Wald test is: reject H_0 if $|W| > z_{\alpha/2}$ where $W = \hat{\delta}/\hat{se}$.

(1b)

$$W = \frac{\overline{X} - \overline{Y}}{\sqrt{\frac{\overline{X}}{n} + \frac{\overline{Y}}{m}}} = \frac{8.5 - 7.5}{\sqrt{\frac{8.5}{100} + \frac{7.5}{100}}} = \frac{10(8.5 - 7.5)}{\sqrt{16}} = 2.5$$

so

$$p - value = \mathbb{P}(|Z| > 2.5) = .012.$$

(1c) Under H_1 the likelihood is

$$\mathcal{L}(\lambda, \nu) = e^{-n\lambda}\lambda^{n\overline{X}}e^{-m\nu}\nu^{m\overline{Y}}.$$

Under H_0, let $\gamma \equiv \lambda = \nu$. Then

$$\mathcal{L}(\gamma) = e^{-(n+m)\gamma}\gamma^{(n+m)\hat{\gamma}}$$

where

$$\hat{\gamma} = \frac{\sum_{i=1}^n X_i + \sum_{i=1}^m Y_i}{n+m} = \alpha\hat{\lambda} + (1-\alpha)\hat{\gamma}$$

is the maximizer, and $\alpha = n/(n+m)$. Then

$$\lambda = 2\log\left(\frac{\mathcal{L}(\hat{\lambda}, \hat{\nu})}{\mathcal{L}(\hat{\gamma})}\right) = \left(\frac{\overline{X}}{\alpha\overline{X} + (1-\alpha)\overline{Y}}\right)^{n\overline{X}}\left(\frac{\overline{Y}}{\alpha\overline{X} + (1-\alpha)\overline{Y}}\right)^{m\overline{Y}}.$$

Reject if $\lambda > \chi^2_{1,\alpha}$.

(2a) $f(p|X) \propto p$ so $f(p|X) = 2p$. The posterior mean is

$$\mathbb{E}(p|X=1) = \int_0^1 pf(p|X=1)dp = \int_0^2 2p^2 dp = 2/3.$$

(2b) We have

$$f(p|x) = \begin{cases} 2p & \text{if } x = 1 \\ 2(1-p) & \text{if } x = 0. \end{cases}$$

So

$$1 - \alpha = \int_0^a f(p|1)dp = a^2$$

implies $a(1) = \sqrt{1-\alpha}$. Also,

$$1 - \alpha = \int_0^a f(p|0)dp = a(2-a)$$

implies $a(0) = 1 - \sqrt{\alpha}$. Hence,

$$a(X) = X\sqrt{1-\alpha} + (1-X)(1-\sqrt{\alpha}).$$

(c) Note that $1 - \sqrt{\alpha} < sqrt1 - \alpha$. So,

$$\mathbb{P}(p \in A) = \mathbb{P}(p \le X\sqrt{1-\alpha} + (1-X)(1-\sqrt{\alpha}))$$

which is

$$\mathbb{P}(p \in A) = \begin{cases} 0 & \text{if } p > \sqrt{1-\alpha} \\ p & \text{if } 1 - \sqrt{\alpha} < p < \sqrt{1-\alpha} \\ 1 & \text{if } p < 1 - \sqrt{\alpha}. \end{cases}$$

(3a) Let $\hat{\theta}$ be admissible. Suppose $\hat{\theta} \notin \mathcal{C}$. We will derive a contradiction. Since $\hat{\theta} \notin \mathcal{C}$. Then there exists $\tilde{\theta} \in \mathcal{C}$ that is better than $\hat{\theta}$. This contradicts the fact that $\hat{\theta}$ is admissible.

(3b) The risk is $R(p, \hat{p}) = p(1-p)/(np(1-p)) = 1/n$ which is constant. It thus suffices to show that $\hat{p}$ is a Bayes rule. The posterior risk is

$$r = \int_0^2 \frac{(\hat{p} - p)^2}{p(1-p)} f(p|x^n)dp.$$

Setting $dr/d\hat{p} = 0$ gives

$$\hat{p} = frac \int_0^1 \frac{f(p|x^n)}{1-p} dp \int_0^1 \frac{f(p|x^n)}{p(1-p)} dp.$$

Under the prior $f(p) = 1$ we get

$$\begin{aligned}
\hat{p} &= frac \int_0^1 \frac{f(p|x^n)}{1-p} dp \int_0^1 \frac{f(p|x^n)}{p(1-p)} dp \\
&= \frac{\int_0^1 p^X (1-p)^{n-X-1} dp}{\int_0^1 p^{X-1} (1-p)^{n-X-1} dp} \\
&= \frac{\frac{\Gamma(X+1)\Gamma(n-X)}{\Gamma(n+1)} \int_0^1 \frac{\Gamma(n+1)}{\Gamma(X+1)\Gamma(n-X)} p^{X+1-1} (1-p)^{n-X-1} dp}{\frac{\Gamma(X)\Gamma(n-X)}{\Gamma(n)} \int_0^1 \frac{\Gamma(n)}{\Gamma(X)\Gamma(n-X)} p^{X-1} (1-p)^{n-X-1} dp} \\
&= \frac{\frac{\Gamma(X+1)\Gamma(n-X)}{\Gamma(n+1)}}{\frac{\Gamma(X)\Gamma(n-X)}{\Gamma(n)}} \\
&= \frac{X \frac{\Gamma(X)\Gamma(n-X)}{n\Gamma(n)}}{\frac{\Gamma(X)\Gamma(n-X)}{\Gamma(n)}} \\
&= \frac{X}{n}.
\end{aligned}$$

So $\hat{P}$ is the Bayes estimator and hence is minimax.