Solutions: Pratice Test 3

(1a) Reject if $|W| > z_{\alpha/2}$ where $W = (\hat{p} - p_0)/\hat{se}$ and $\hat{se} = \sqrt{\hat{p}(1-\hat{p})/n}$.
(1b) The power is

$$\begin{aligned}
\beta(p) &= \mathbb{P}(W < -z_{\alpha/2}) + \mathbb{P}(W > z_{\alpha/2}) \\
&= \mathbb{P}\left(\frac{\hat{p}-p_0}{\hat{se}} < -z_{\alpha/2} + \frac{p-p_0}{\hat{se}}\right) + \mathbb{P}\left(\frac{\hat{p}-p_0}{\hat{se}} > z_{\alpha/2} + \frac{p-p_0}{\hat{se}}\right) \\
&\approx \mathbb{P}\left(Z < -z_{\alpha/2} + \frac{p-p_0}{\hat{se}}\right) + \mathbb{P}\left(Z > z_{\alpha/2} + \frac{p-p_0}{\hat{se}}\right) \\
&= \Phi\left(Z < -z_{\alpha/2} + \frac{p-p_0}{\hat{se}}\right) + 1 - \Phi\left(Z > z_{\alpha/2} + \frac{p-p_0}{\hat{se}}\right).
\end{aligned}$$

(2) $\hat{\mu} = \overline{X}$, $\hat{\sigma}^2 = n^{-1}\sum_{i=1}^n (X_i - \overline{X})^2$,

$$\mathcal{L}(\mu,\sigma) = \sigma^{-n}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (X_i-\mu)^2\right\},$$

and

$$\mathcal{L}(\hat{\mu},\hat{\sigma}) = \hat{\sigma}^{-n}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (X_i-\hat{\mu})^2\right\} = \hat{\sigma}^{-n}e^{-n/2}.$$

The restricted mle for μ is obtained by maximizing $\mathcal{L}(\mu,1)$ yielding $\hat{\mu}_0 = \overline{X}$. Hence,

$$\lambda = 2\log\frac{\mathcal{L}(\hat{\mu},\hat{\sigma})}{\mathcal{L}(\hat{\mu}_0,1)} = 2n\log\frac{1}{\hat{\sigma}} + n(\hat{\sigma}^2 - 1).$$

Reject if $\lambda > \chi^2_{1,\alpha}$.

(3) This was a homework problem.

(4) The likelihood is

$$\mathcal{L}(\beta) = \beta^{-n}e^{-n\overline{X}/\beta},$$

$$\ell(\beta) = -n\log\beta - \frac{n\overline{X}}{\beta}$$

so $\hat{\beta} = \overline{X}$. Thus,

$$\lambda = 2\log\frac{\mathcal{L}(\hat{\mu},\hat{\sigma})}{\mathcal{L}(\hat{\mu}_0,1)} = 2\ell(\hat{\beta}) - 2\ell(\beta_0) = 2n\log\left(\frac{\beta_0}{\overline{X}}\right) + 2n\left(\frac{\overline{X}}{\beta_0} - 1\right).$$

Reject if $\lambda > \chi^2_{1,\alpha}$.

(4b) Under H_0, $X_i \sim \text{Exp}(\beta_0)$. Hence, $T = \sum_{i=1}^n X_i \sim \text{Gamma}(n,\beta_0)$. Let F denote the cdf of a $\text{Gamma}(n,\beta_0)$. Let $R = (0, F^{-1}(\alpha/2)) \cup (F^{-1}(1-\alpha/2),\infty)$. Reject if $T \in R$.

(4c) The power is $\mathbb{P}(U \in R)$ where $U \sim \text{Gamma}(n,\beta)$.

(5a) Reject H_0 if $|W| > z_{\alpha/2}$ where $W = \sqrt{n}(\overline{X} - 0)/\sigma$.

(5b) Reject H_0 if $|W| > z_{\alpha/2}$ where $W = \sqrt{n}(\overline{X} - 0)/\hat{\sigma}$.

(5c) (i) Draw $X_1^*,\ldots,X_n^* \sim \hat{F}$. (ii) Compute $\hat{\mu}^* = n^{-1}\sum_{i=1}^n X_i^*$. (iii) Repeat (i)-(ii) B times to get $\hat{\mu}_1^*,\ldots,\hat{\mu}_B^*$. (iv) Let $\hat{se} = \sqrt{B^{-1}\sum_{i=1}(\hat{\mu}_i^* - \overline{\mu}^*)^2}$. (v) Let $W = (\hat{\mu} - 0)/\hat{se}$. (vi) Reject if $|W| > z_{\alpha/2}$.

(5d) Same but replace (i) with: Draw $X_1^*,\ldots,X_n^* \sim N(\hat{\mu},\hat{\sigma}^2)$.

(6) Under H_0, $(\hat{\theta} - \theta_0)/\hat{se}(\hat{\theta}) \approx N(0,1)$. By the delta method, $U \approx N(0,1)$. Hence,

$$\mathbb{P}(|U| > z_{\alpha/2}) \to \mathbb{P}(|Z| > z_{\alpha/2}) = \alpha.$$

(7) The observed test statistic is $|2-3| = 1$. The permutations are:

permutation	T
2,1,5	1
2,5,1	1
1,2,5	4.5
1,5,2	4.5
5,1,2	3.5
5,2,1	3.5

Therefore,

$$p - value = \mathbb{P}_0(T > 1) = 4/6.$$

(8a) $\mathcal{L}(\theta) = e^{-n(\overline{X}-\theta)^2/2}$. So

$$\lambda = 2\log\left(\frac{\mathcal{L}(\hat{\theta})}{\mathcal{L}(0)}\right) = n\overline{X}^2.$$

Reject H_0 if $\lambda > \chi^2_{1,\alpha}$.

(b) Under H_0, $\sqrt{n}(\overline{X} - 0) \sim N(0,1)$. So $\lambda \sim (N(0,1))^2 \equiv \chi^2_1$ so the χ^2 approximation to the distribution of λ is exact in this case.

(9) They are all false.

(10a) The Fisher information matrix is

$$I(\mu,\sigma) = \left[\begin{array}{cc} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{array}\right].$$

So

$$f(\mu,\sigma) \propto \sqrt{\det(I)} = \sqrt{\frac{2}{\sigma^4}} \propto \frac{1}{\sigma^2}.$$

(10b)

$$\begin{aligned} f(\mu,\sigma|x^n) &\propto \mathcal{L}(\mu,\sigma)f(\mu,\sigma) \propto \sigma^{-n}\exp\left\{-\frac{n-1}{2\sigma^2}S^2\right\}\exp\left\{-\frac{n}{2\sigma^2}(\overline{X}-\mu)^2\right\}\sigma^{-2} \\ &= \sigma^{-n-2}\exp\left\{-\frac{n-1}{2\sigma^2}S^2\right\}\exp\left\{-\frac{n}{2\sigma^2}(\overline{X}-\mu)^2\right\}. \end{aligned}$$

(10c) The posterior is

$$f(\mu|x^n) = \int_0^\infty f(\mu,\sigma|x^n)d\sigma \propto \int_0^\infty \sigma^{-n-2}\exp\left\{-\frac{n-1}{2\sigma^2}S^2\right\}\exp\left\{-\frac{n}{2\sigma^2}(\overline{X}-\mu)^2\right\}d\sigma.$$

This turns out to be a t-distribution but we didn't cover the necessary material in class to finish this.

(11) The Bayes estimator under squared error loss is the posterior mean. Also, the risk is the MSE which is $bias^2 + Var$.

(a) The posterior is

$$f(p|x) \propto p^X(1-p)^{n-X}p^{\alpha-1}(1-p)^{\beta-1} = p^{X+\alpha-1}(1-p)^{n-X+\beta-1}$$

so $p|x \sim \text{Beta}(\alpha+X, \beta+n-X)$. Thus,

$$\overline{p} = \mathbb{E}(p|X) = \frac{\alpha+X}{n+\alpha+\beta}.$$

Now,

$$\begin{aligned}
\mathbb{E}(\overline{p}) &= \frac{\alpha+\mathbb{E}(X)}{n+\alpha+\beta} = \frac{\alpha+np}{n+\alpha+\beta} \\
bias(\overline{p} &= \frac{\alpha+np}{n+\alpha+\beta} - p = \frac{\alpha(1-p)-\beta p}{n+\alpha+\beta} \\
\mathbb{V}(\overline{p}) &= \frac{np(1-p)}{n+\alpha+\beta}^2 \\
R(p,\overline{p}) &= \left(\frac{\alpha(1-p)-\beta p}{n+\alpha+\beta}\right)^2 + \frac{np(1-p)}{n+\alpha+\beta}^2 \\
r(\pi,\overline{p}) &= \int R(p,\overline{p})\pi(p)dp = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int R(p,\overline{p})p^{\alpha-1}(1-p)^{\beta-1} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(n+\alpha+\beta)\Gamma(\alpha+\beta+2)} \\
&= \frac{\alpha\beta}{(n+\alpha+\beta)(\alpha+\beta+1)(\alpha+\beta)}
\end{aligned}$$

where we used the fact that $\Gamma(x+1) = x\Gamma(x)$.

(b) The posterior is

$$f(\lambda|X) = e^{-n\lambda}\lambda^{n\overline{X}}\frac{1}{\Gamma(\alpha)\beta^\alpha}\lambda^{\alpha-1}e^{-\lambda/\beta} \propto \lambda^{a-1}e^{-\lambda/b}$$

where $a = X + \alpha$, $b = \beta/(\beta+1)$. So $\lambda|X \sim \text{gamma}(a,b)$. The posterior mean is

$$\overline{\lambda} = \mathbb{E}(\lambda|X) = ab = \frac{X+\alpha}{1+\frac{1}{\beta}}.$$

Thus

$$\text{bias} = \frac{\lambda+\alpha}{1+\frac{1}{\beta}} - \lambda = \frac{\alpha-\lambda/\beta}{1+1/\beta}$$

$$
\begin{aligned}
\text{var} &= \frac{\lambda}{1+1/\beta}^{2} \\
R(\lambda, \overline{\lambda}) &= \frac{\beta^2\alpha^2 - 2\beta\alpha\lambda + \lambda^2 + \lambda\beta^2}{(\beta+1)^2} \\
r(\pi, \overline{\lambda}) &= \int_0^\infty R(\lambda, \overline{\lambda}) \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} e^{-\lambda/\beta} d\lambda \\
&= \frac{1}{\Gamma(\alpha)} \frac{\beta^{2-\alpha}\beta^\alpha \Gamma(\alpha+1)}{1+\beta} \\
&= \frac{\alpha\beta^{2-\alpha}\beta^\alpha}{1+\beta}.
\end{aligned}
$$

(c) $\theta | X \sim N(\overline{\theta}, \tau^2)$ where

$$
\overline{\theta} = \frac{\frac{X}{\sigma^2} + \frac{a}{b^2}}{\frac{1}{\sigma^2} + \frac{1}{b^2}}, \quad \tau^2 = \frac{1}{\frac{1}{\sigma^2} + \frac{1}{b^2}}.
$$

So,

$$
\begin{aligned}
\text{bias} &= \frac{\frac{\theta}{\sigma^2} + \frac{a}{b^2}}{\frac{1}{\sigma^2} + \frac{1}{b^2}} - \theta = \frac{\frac{a-\theta}{b^2}}{\frac{1}{\sigma^2} + \frac{1}{b^2}} \\
\text{var} &= \frac{\frac{1}{\sigma^2}}{\frac{1}{\sigma^2} + \frac{1}{b^2}} \\
R(\theta, \overline{\theta}) &= \frac{\frac{(a-\theta)^2}{b^4} + \frac{1}{\sigma^2}}{\left(\frac{1}{\sigma^2} + \frac{1}{b^2}\right)^2} \\
r(\pi, \theta) &= \frac{\mathbb{E}_\pi\left(\frac{(a-\theta)^2}{b^4}\right) + \frac{1}{\sigma^2}}{\left(\frac{1}{\sigma^2} + \frac{1}{b^2}\right)^2} \\
&= \frac{1}{\frac{1}{b^2} + \frac{1}{\sigma^2}}.
\end{aligned}
$$

(12) Let R denote the risk function and let $\tilde{R}$ denote risk under squared error loss. Thus, $R(\theta, \hat{\theta}) = \sigma^{-2}\tilde{R}(\theta, \hat{\theta})$. We know that $\overline{X}$ is admissible under

squared error loss by Theorem 13.21. This implies that $\overline{X}$ is admissible under the given loss function. For if not, then there exists $\hat{\theta}$ such that $R(\theta,\hat{\theta}) \leq R(\theta,\overline{X})$ with strict inequality at at least one θ. But that imples that $\sigma^{-2}R(\theta,\hat{\theta}) \leq \sigma^{-2}R(\theta,\overline{X})$ and so $\tilde{R}(\theta,\hat{\theta}) \leq \tilde{R}(\theta,\overline{X})$ contradicting the admissibility of $\overline{X}$ under squared error loss. The risk of $\overline{X}$ is $\mathbb{E}(\overline{X}-\theta)^2/\sigma^2 = 1/n$. Since $\overline{X}$ is admissible with constant risk, it is minimax by Theorem 13.22.

(13) The loss is

$$L(\theta,\hat{\theta}) = \begin{cases} 1 & \text{if } \theta = \hat{\theta} \\ 0 & \text{if } \theta \neq \hat{\theta}. \end{cases}$$

The posterior risk is

$$r(\hat{\theta}|x) = \sum_{j=1}^{k} L(\hat{\theta},\theta_j) f(\theta_j|x) = \sum_{j:\ \hat{\theta}\neq\theta_j} f(\theta_j|x) = 1 - f(\hat{\theta}|x).$$

This is minimized by maximizing $f(\hat{\theta}|x)$, that is, take $\hat{\theta}(x)$ to maximize $f(\theta|x)$.

(14) Let $\hat{\sigma}^2 = bS^2$. The risk is

$$\begin{aligned} R(\sigma^2,\hat{\sigma}^2) &= \mathbb{E}\left(\frac{bS^2}{\sigma^2} - 1 - \log\frac{bS^2}{\sigma^2}\right) \\ &= b\left(\frac{\mathbb{E}(S^2)}{\sigma^2}\right) - 1 - \log b - \mathbb{E}\left(\log\frac{S^2}{\sigma^2}\right) \\ &= b - \log b - 1 - \mathbb{E}\left(\log\frac{S^2}{\sigma^2}\right). \end{aligned}$$

Note that the last term does not depend on b. Take the derivative with respect to b. The minimum occurs at $b = 1$.

(15) The risk of $\hat{p}$ is 1. Suppose $p'(x)$ is another estimator. Then $B = \{x:\ p'(x) > 0\}$ is not empty. Let $b = \min\{p'(x):\ x \in B\}$. Note that $b > 0$.

When $p < (b/2)$ and $x \in B$,

$$L(p, p') = \left(1 - \frac{p'(x)}{p}\right)^2 > 1.$$

Also, $L(p, p') = 1$ for $x \in B^c$. Hence,

$$\begin{aligned} R(p, p') &= \sum_{x=0}^{n} L(p, p'(x)) f(x|p) = \sum_{x \in B}^{n} L(p, p'(x)) f(x|p) + \sum_{x \notin B}^{n} L(p, p'(x)) f(x|p) \\ &> \sum_{x \in B}^{n} f(x|p) + \sum_{x \notin B}^{n} f(x|p) = 1. \end{aligned}$$