Solutions to Practice Test 1

(1) The CDF is

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x/2 & 0 \le x < 1 \\ 1/2 & 1 \le x < 3 \\ (1/2) + (x-3)/2 & 3 \le x < 4 \\ 1 & x > 4. \end{cases}$$

(2) Y takes only two values, 0 and 1. $\mathbb{P}(Y = 1) = \mathbb{P}(X \in A) = \int_A f(x)dx \equiv p_A$ $\mathbb{P}(Y = 0) = \mathbb{P}(X \in A^c) = \int_{A^c} f(x)dx 1 - p_A$. So

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ 1 - p_A & 0 \le y < 1 \\ 1 & y \ge 1. \end{cases}$$

(3) First find the distribution of $V = \log(X)$:

$$F_V(v) = \mathbb{P}(V \le v) = \mathbb{P}(\log X \le v) = \mathbb{P}(X \le e^v) = \int_0^{e^v} f_X(x)dx = \int_0^{e^v} dx = e^v.$$

Hence,

$$f_V(v) = \begin{cases} e^v & v < 0 \\ 0 & v \ge 0. \end{cases}$$

Similarly for $W = \log(Y)$. The joint density of (V, W) is

$$f(v, w) = \begin{cases} e^{v+w} & v, w < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Now $Z = V + W$ and so $-\infty < Z < 0$. Now $A_z = \{(v, w) : \ v + w \le z\}$ is the complement of the triangle with vertices $(z, 0), (0, 0), (0, z)$ (with $z < 0$). Hence, for $z < 0$,

$$\begin{aligned} 1 - F_Z(z) &= \mathbb{P}(Z \le z) = \mathbb{P}(V + W \le z) \\ &= \int_z^0 \int_{z-w}^0 f(v, w) dv dw \end{aligned}$$

$$
\begin{aligned}
&= \int_z^0 e^v \int_{z-v}^0 e^w dw dv \\
&= \int_z^0 e^v \left(1 - e^{z-v}\right) dv \\
&= 1 - e^z + ze^z.
\end{aligned}
$$

Thus,

$$f_Z(z) = -ze^z$$

for $z < 0$.

(4) Check the three axioms:

$$Q(A) = \mathbb{P}(A|C) = \frac{\mathbb{P}(AC)}{\mathbb{P}(C)} \geq 0$$

$$Q(\Omega) = \mathbb{P}(\Omega|C) = \frac{\mathbb{P}(\Omega \bigcap C)}{\mathbb{P}(C)} = \frac{\mathbb{P}(C)}{\mathbb{P}(C)} = 1$$

Suppose that $A_1, A_2, \ldots$ are disjoint. Then $A_1 \bigcap C, A_2 \bigcap C, \ldots$ are also disjoint and hence

$$
\begin{aligned}
Q\left(\bigcup_{i=1}^{\infty} A_i\right) &= \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i \,\middle|\, C\right) = \frac{\mathbb{P}\Big(\big(\bigcup_{i=1}^{\infty} A_i\big) \bigcap C\Big)}{\mathbb{P}(C)} \\
&= \frac{\mathbb{P}\Big(\bigcup_{i=1}^{\infty}\big(A_i \bigcap C\big)\Big)}{\mathbb{P}(C)} \\
&= \frac{\sum_{i=1}^{\infty} \mathbb{P}\Big(A_i \bigcap C\Big)}{\mathbb{P}(C)} \\
&= \sum_{i=1}^{\infty} \mathbb{P}\left(A_i \,\middle|\, C\right) \\
&= \sum_{i=1}^{\infty} Q(A_i).
\end{aligned}
$$

(5) $\mathbb{P}(Y = 0) = \mathbb{P}(X < (1/2)) = \int_0^{1/2} f(x)dx = \int_0^{1/2} 2xdx = 1/4$. For $y > 1$,

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y < y) = \mathbb{P}(Y = 0) + \mathbb{P}(1 < Y < y) = \frac{1}{4} + \mathbb{P}(1 < 2X < y) \\ &= \frac{1}{4} + \mathbb{P}\left(\frac{1}{2} < X < \frac{y}{2}1\right) \\ &= \frac{1}{4} + \int_{1/2}^{y/2} 2xdx \\ &= \frac{1}{4} + \frac{1}{4}\left(y^2 - 1\right). \end{aligned}$$

Therefore,

$$F_Y(y) = \begin{cases} 0 & x < 0 \\ 1/4 & 0 \leq y \leq 1 \\ \frac{1}{4} + \frac{1}{4}(y^2 - 1) & 1 < y \leq 2 \\ 1 & y > 2. \end{cases}$$

(6) An inspired guess is that $Y = cX$ for some c. Then

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(cX \leq y) = \mathbb{P}\left(X \leq \frac{y}{c}\right) = 2\int_0^{y/c} e^{-2x}dx = 1 - e^{-2y/c}$$

and $f_Y(y) = (2/c)e^{-2y/c}$. So take $c = 2$, i.e. $Y = 2X$.

(7a) To find c:

$$1 = \int_0^1 \int_0^1 f(x,y)dxdy = c\int_0^1 \int_0^1 (x+y)dxdy = c$$

so $c = 1$.

(7b) First we need $f_X(x)$:

$$f_X(x) = \int_0^1 f(x,y)dy = x + \frac{1}{2}.$$

So

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{x+y}{x+\frac{1}{2}}.$$

(7c)

$$\mathbb{P}(Y > (1/2)|X = 1) = \int_{(1/2)}^{1} f(y|1)dy = \int_{(1/2)}^{1} \frac{1+y}{1+\frac{1}{2}}dy = \frac{7}{12}.$$

(7d)

$$\mathbb{P}(Y > (1/2)|X < (1/2)) = \frac{\mathbb{P}(Y > (1/2), X < (1/2))}{\mathbb{P}(X < (1/2))} == \frac{\int_0^{1/2}\int_{1/2}^{1} f(x,y)dydx}{\int_0^{1/2} f_X(x)dx} = \frac{\frac{1}{4}}{\frac{3}{8}} = \frac{2}{3}.$$

(8)

$$\begin{aligned} \mathsf{Cov}(X,Y) &= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \mathbb{E}\mathbb{E}(XY|X) - \mathbb{E}(X)\mathbb{E}\mathbb{E}(Y|X) \\ &= \mathbb{E}X\mathbb{E}(Y|X) - \mathbb{E}(X)\mathbb{E}\mathbb{E}(Y|X) = \mathbb{E}X^2 - \mathbb{E}(X)\mathbb{E}(X) \\ &= \mathbb{E}X^2 - (\mathbb{E}(X))^2 = \mathbb{V}(X). \end{aligned}$$

(9a) $\mathbb{P}(Y = 0) = 1-b > 0$ and $\mathbb{P}(Z = 0) = a > 0$ but $\mathbb{P}(Y = 0, Z = 0) = 0$ so they are not independent.

(9b) The joint distribution is

	$Z = 0$	$Z = 1$
$Y = 0$	0	1-b
$Y = 1$	a	b-a

The conditins distributions for Y given Z are

y	$\mathbb{P}(Y = y\|Z = 0)$	$\mathbb{P}(Y = y\|Z = 1)$
0	0	$\frac{1-b}{1-a}$
1	1	$\frac{b}{1-a}$

So

$$\mathbb{E}(Y|Z = 0) = (0 \times 0) + (1 \times 1) = 1$$

and

$$\mathbb{E}(Y|Z = 1) = \left(0 \times \frac{1-b}{1-a}\right) + \left(1 \times \frac{b}{1-a}\right) = \frac{b}{1-a}.$$

Therefore,

$$\mathbb{E}(Y|Z) = I(Z = 0) + I(Z = 1)\frac{b}{1-a}.$$