Solutions: Test 2

(1) Let $W = X + \sum_{i=1}^n Y_i$. Then We have

$$\mathbb{E}(W) = np + n\lambda = n(p+\lambda)$$

and

$$\mathbb{V}(W) = np(1-p) + n\lambda = n(p(1-p)+\lambda).$$

By the CLT $W \approx N(n(p+\lambda), n(p(1-p)+\lambda))$. So

$$\mathbb{P}(W > c) \approx \mathbb{P}\left(Z > \frac{c - n(p+\lambda)}{\sqrt{n(p(1-p)+\lambda)}}\right) = .02$$

which implies that

$$\frac{c - n(p+\lambda)}{\sqrt{n(p(1-p)+\lambda)}} = z_{.02}$$

and hence

$$c = z_{.02}\sqrt{n(p(1-p)+\lambda)} + n(p+\lambda).$$

(2a) Let $S = \sum_{i=1}^n X_i$ and $T = \sum_{i=1}^n Y_i$. (a) $\mathcal{L}(\lambda,\nu) \propto e^{-n\lambda}\lambda^S e^{-m\nu}\lambda^T$.

(2b)

$$\ell(\lambda,\nu) = -n\lambda + S\log(\lambda) - m\nu + T\log(\nu)$$

So

$$\begin{aligned} \frac{\partial \ell}{\partial \lambda} &= -n + \frac{S}{\lambda} \\ \frac{\partial \ell}{\partial \nu} &= -m + \frac{T}{\nu} \end{aligned}$$

hence

$$\hat{\lambda} = \frac{S}{n} \quad \text{and} \quad \hat{\nu} = \frac{T}{m}.$$

(2c) The matrix of second derivatives is

$$H = \begin{pmatrix} -\frac{\sum_{i=1}^n X_i}{\lambda^2} & 0 \\ 0 & -\frac{\sum_{i=1}^m Y_i}{\nu^2} \end{pmatrix}$$

and so

$$I(\lambda,\nu) = \begin{pmatrix} \frac{n}{\lambda} & 0 \\ 0 & \frac{m}{\nu} \end{pmatrix}$$

(2d) Let $g(\lambda,\nu) = \lambda + \mu$. Then $\nabla^T = (1 \;\; 1)$ and

$$\mathsf{se}(\hat\psi) = \sqrt{\nabla^T I^{-1} \nabla} = \sqrt{(1 \;\; 1) \begin{pmatrix} \frac{\lambda}{n} & 0 \\ 0 & \frac{\nu}{m} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \sqrt{\frac{\lambda}{n} + \frac{\nu}{m}}$$

so the confidence interval is

$$\frac{S}{n} + \frac{T}{m} \pm 1.64 \sqrt{\frac{\hat\lambda}{n} + \frac{\hat\nu}{m}}.$$

(3a) Let $W = \max\{|X_1|, \ldots, |X_n|\}$. Then,

$$\mathcal{L}(\theta) = \begin{cases} 0 & \text{if } \theta < W \\ \left(\frac{1}{2\theta}\right)^n & \theta \geq W. \end{cases}$$

This is maximized at $\theta = W$.

(3b) Let $Y_i = |X_i|$. Then $Y_i \sum \text{Uniform}(0,\theta)$. Since $\hat\theta \leq \theta$,

$$\mathbb{P}(|\hat\theta - \theta| > \epsilon) = \mathbb{P}(\hat\theta < \theta - \epsilon) = \mathbb{P}(\max\{Y_1, \ldots, Y_n\} \leq \theta - \epsilon) = \prod_{i=1}^n \mathbb{P}(Y_i \leq \theta - \epsilon) = \left(\frac{\theta - \epsilon}{\theta}\right)^n \to 0.$$

(3c) $\mathbb{E}_\theta(X_i) = 0$ which does not involve θ. Instead, we could use the first moment of Y_i which is $E_\theta(Y_i) = \theta/2$ leading to

$$\frac{\hat\theta}{2} = \frac{1}{n} \sum_{i=1}^n Y_i$$

so $\hat\theta = 2\overline{Y}_n$. There are other possibilities such as using $\mathbb{E}(X_i^2)$ or $\mathbb{V}(X_i)$.

(3d)

$$\psi = \mathbb{P}(X < 1) = \int_{-\theta}^{1} f(x;\theta)dx = \frac{1+\theta}{2\theta}$$

so

$$\hat{\psi} = \frac{1+\hat{\theta}}{2\hat{\theta}}.$$

We can write

$$\psi = \mathbb{E}I(X < 1) = \int I(x<1)dF(x)$$

so the plug in estimator

$$\tilde{\psi} = \int I(x<1)d\hat{F}(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i < 1).$$

(3e) Nonparametric bootstrap:
(1) $X_1^*, \ldots, X_n^* \sim \hat{F}$
(2) $\tilde{\psi}^* = \frac{1}{n}\sum_{i=1}^{n} I(X_i^* < 1)$
(3) repeat to get $\tilde{\psi}_1^*, \ldots, \tilde{\psi}_B^*$
(4)

$$\hat{\mathsf{se}}_{boot} = \sqrt{\frac{1}{B}\sum_{j=1}^{B}(\tilde{\psi}_j^* - \overline{\psi}^*)^2}$$

Parametric bootstrap is the same except in (1) we sample from Uniform$(-\hat{\theta}, \hat{\theta})$.