Homework 1: Solutions

1. Claim 1: $B_1, B_2, \ldots$ are disjoint. Proof: Let $\omega \in B_i$. If $\omega' \in B_j$ for $j < i$ then $\omega' \in A_j$ and hence $\omega' \notin B_i$. Therefore, $B_i B_j = \emptyset$. If $\omega' \in B_j$ for $j > i$ then $\omega' \in A_j$ and $\omega' \notin A_{j-1}$. Since $A_i \subset A_{j-1}$ then $\omega' \notin A_i$. Hence, $\omega' \notin B_i$. Therefore, $B_i B_j = \emptyset$.

Claim 2: $\bigcup_{1=1}^n A_i = \bigcup_{1=1}^n B_i$. Proof: $\omega \in \bigcup_{1=1}^n A_i \implies \omega \in A_i$ for some i $\implies$. Let i_0 be the smallest i such that $\omega \in A_i$. If $i_0 = 1$ then $\omega \in B_1$. If $i_0 > 1$ then $\omega \in A_{i_0}$ but $\omega \notin A_i$ for $i < i_0$. Hence, $\omega \in B_i$. Hence $\omega \in \bigcup_{i=1}^n B_i$. For the reverse direction, suppose that $\omega \in \bigcup_{i=1}^n B_i$. Then $\omega \in B_i$ for some i. Hence, $\omega \in A_i$ and therefore, $\omega \in \bigcup_{1=1}^n A_i$.

Claim 3: $\bigcup_{1=1}^\infty A_i = \bigcup_{1=1}^\infty B_i$. The proof is the same as for Claim 2.

The monotone decreasing case. Let $A = \lim_n A_n = \bigcap_{n=1}^\infty A_n$. Then $A^c = (\bigcap_n A_n)^c = \bigcup_n A_n^c$. Since the A_n are monotone decreasing, the A_n^c are monotone increasing. Therefore $1 - \mathbb{P}(A) = \mathbb{P}(A^c) = \lim_{n\to\infty} \mathbb{P}(A_n^c) = \lim_{n\to\infty}(1 - \mathbb{P}(A_n)) = 1 - \lim_{n\to\infty} \mathbb{P}(A_n)$. Hence, $\mathbb{P}(A) = \lim_{n\to\infty} \mathbb{P}(A_n)$.

2. (i) $1 = \mathbb{P}(\Omega) = \mathbb{P}(\Omega \bigcup \emptyset) = \mathbb{P}(\Omega) + \mathbb{P}(\emptyset) = 1 + \mathbb{P}(\emptyset)$. Therefore $\mathbb{P}(\emptyset) = 0$.

(ii) Let $C = B - A$. Then $A \bigcap C = \emptyset$ and $A \bigcup C = B$. So $\mathbb{P}(B) = \mathbb{P}(A \bigcup C) = \mathbb{P}(A) + \mathbb{P}(C) \geq \mathbb{P}(A)$.

(iii) Since $A \subset \Omega$, $\mathbb{P}(A) \leq \mathbb{P}(\Omega) = 1$ by (ii). Also, $\emptyset \subset A$ so $0 = \mathbb{P}(\emptyset) \leq \mathbb{P}(A)$.

(iv) Since $\Omega = A \bigcup A^c$ and A and A^c are disjoint, $1 = \mathbb{P}(\Omega) = \mathbb{P}(A) + \mathbb{P}(A^c)$ and hence $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

(v) Let $A_1 = A$, $A_2 = B$, $A_i \emptyset$ for $i \geq 3$. Then $A_1, A_2, \ldots$ are disjoint. Hence, $\mathbb{P}(A \bigcup B) = \mathbb{P}(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mathbb{P}(A_i) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(A_3) + \mathbb{P}(A_4) + \cdots = \mathbb{P}(A) + \mathbb{P}(B) + 0 + 0 + \cdots = \mathbb{P}(A) + \mathbb{P}(B)$.

3. Let $B_n = \bigcup_{i=n}^{\infty} A_i$ and $C_n = \bigcap_{i=n}^{\infty} A_i$.

(3a) $\omega \in C_n \Longrightarrow \omega \in \bigcap_{i=n}^{\infty} A_i \Longrightarrow \omega \in \bigcap_{i=n+1}^{\infty} A_i \Longrightarrow \omega \in C_{n+1}$. Therefore, $C_n \subset C_{n+1}$.

$\omega \in B_n \Longrightarrow \omega \in \bigcup_{i=n}^{\infty} A_i \Longrightarrow \omega \in A_{n-1} \bigcup (\bigcup_{i=n}^{\infty} A_i) \Longrightarrow \omega \in \bigcup_{i=n-1}^{\infty} A_i \Longrightarrow \omega \in B_{n-1}$. Therefore, $B_n \subset B_{n-1}$.

(3b) Say that $\omega \in A_n$ in£nitely often (i.o.) if ω belongs to an in£nite number of the events $A_1, A_2, \ldots$. If $\omega \in A_n$ i.o. then $\omega \in \bigcup_{i=n}^{\infty} A_i$ for all $n \Longrightarrow \omega \in B_n$ for all $n \Longrightarrow \omega \in \bigcap_{n=1}^{\infty} B_n$.

If $\omega \notin A_n$ i.o. then there exists n such that $\omega \notin A_j$ for all $j > n \Longrightarrow \omega \notin \bigcup_{i=n+1}^{\infty} \Longrightarrow \omega \notin B_{n+1} \Longrightarrow \omega \notin \bigcap_{n=1}^{\infty} B_n$. Thus, $\omega \in \bigcap_{n=1}^{\infty} B_n \Longrightarrow$ that $\omega \in A_n$i.o..

(3c) Say that $\omega \in A_j$ utlimately (ult.) if ω_j in A_j except for possibly a £nite number of the events. Now, $\omega \in A_j$ ult. $\Longrightarrow \omega \in A_j$ for all $j \geq n$ (for some n) $\Longrightarrow \omega \in \bigcap_{j=n}^{\infty} A_j = C_n \Longrightarrow \omega \in \bigcup_{n=1}^{\infty} C_n$.

$\omega \in \bigcup_{n=1}^{\infty} C_n \Longrightarrow \omega \in C_n$ for some $n \Longrightarrow \omega \in \bigcap_{j=n}^{\infty} A_j \Longrightarrow \omega \in A_j$ for all $j \geq n \Longrightarrow \omega \in A_j$ ult.