Solutions homework 10

(1a) Wald test: reject H_0 when $|W| > z_{\alpha/2}$ where

$$W = \frac{\hat{\mu} - mu_0}{\hat{se}} = \frac{\overline{X} - mu_0}{\frac{\hat{\sigma}}{\sqrt{n}}}.$$

(1b) Take $\hat{\mu}$ to tbe the plug-in estimator which is the sample median. Then compute $\hat{se}$ using the bootstrap. Then use W as above.

(1c) The statistics is

$$\lambda = 2 \log \frac{\mathcal{L}(\hat{\mu}, \hat{\sigma})}{\mathcal{L}(\mu_0, \hat{\sigma}(\mu_0))}$$

where $\hat{\sigma}(\mu_0)$ maximizes $\mathcal{L}(\mu_0, \sigma)$. We get

$$\hat{\sigma}_0^2 = \frac{\sum_{i=1}^n (X_i - \mu_0)^2}{n}$$

and

$$\lambda = 2n \log \left(\frac{\sum_{i=1}^n (X_i - \mu_0)^2}{\sum_{i=1}^n (X_i - \overline{X})^2} \right).$$

Reject when $\lambda > \chi^2_{1,\alpha}$.

(1d) When H_0 is true, $\mathbb{P}(X_i > \mu_0) = 1/2$ and so $\sum_{i=1}^n I(S_i = 1) \sim$ Binomial$(n, 1/2)$. It follows that $\mathbb{P}(|T| > c) \leq \alpha$ by the definition of c.

(2a) The mle is

$$\hat{p} = \left(\frac{X_1}{n}, \ldots, \frac{X_k}{n} \right).$$

Also,

$$\mathcal{L}(p) \propto p_1^{X_1} \cdots p_k^{X_k}.$$

The LR test statistic is

$$2\log\frac{\mathcal{L}(\hat{p})}{\mathcal{L}(p_0)} = 2\log\prod_{j=1}^{k}\left(\frac{\hat{p}_j}{p_{0j}}\right)^{X_j} = 2\sum_{j=1}^{k} X_j \log\frac{X_j}{np_{0j}}.$$

The difference of dimensions is $(k-1) - 0 = k-1$.

(3) We have $X \sim \text{multinomial}(n_1, p)$ and $Y \sim \text{multinomial}(n_2, q)$. We want to test $H_0 : p = q$ versus $H_1 : p \neq q$. Under H_1 the likelihood is

$$\mathcal{L}(p,q) = \prod_{j=1}^{k} p_j^{X_j} \prod_{j=1}^{k} q_j^{Y_j}$$

and $\hat{p} = X/n_1$, $\hat{q} = Y/n_2$. Under H_0, p_j and q_j are equal to a common value; let's call it r_j. So the likelihood under H_0 is

$$\mathcal{L}(r) = \prod_{j=1}^{k} r_j^{X_j} \prod_{j=1}^{k} r_j^{Y_j}.$$

The mle under H_0 is

$$\hat{r} = \left(\frac{X_1+Y_1}{n}, \ldots, \frac{X_k+Y_k}{n}\right)$$

where $n = n_1 + n_2$. The likelihood ratio statistic is

$$\lambda = 2\log\left(\frac{\prod_{j=1}^{k}\hat{p}_j^{X_j}\ \prod_{j=1}^{k}\hat{q}_j^{X_j}}{\prod_{j=1}^{k}\hat{r}_j^{X_j}\ \prod_{j=1}^{k}\hat{r}_j^{X_j}}\right) = 2\sum_{j=1}^{k} X_j\log\left(\frac{X_j}{X_j+Y_j}\frac{n}{n_1}\right) + 2\sum_{j=1}^{k} Y_j\log\left(\frac{Y_j}{X_j+Y_j}\frac{n}{n_2}\right).$$

The difference of dimensions is $\Big[(k-1)+(k-1)\Big] - (k-1) = k-1$. Reject when $\lambda > \chi^2_{k-1,\alpha}$.