Homework 2: Solutions

1. Chapter 2, Problem 5. Ω is the set of finite sequences $\{(\omega_1, \ldots, \omega_k)\}$ such that $\omega_k = H$, exactly one of $\{(\omega_1, \ldots, \omega_{k-1})\}$ is H and the rest are T. Let A be the event that k tosses are required. Then ω is in A if and only if it is of the form

$$\underbrace{\omega_1, \ldots, \omega_{k-1}}_{k-1 \text{ T and one} H} \underbrace{\omega_k}_{H}.$$

Each such sequence has probability $1/2^k$ and there are $\binom{k-1}{1} = k - 1$ such sequences. So the probability is $(k-1)/2^k$.

2. Chapter 2, Problem 6. Suppose there exists a P that is uniform on Ω. Then there is a constant c such that $c = P(\{s_i\})$ for all i. If $c = 0$ then $1 = P(S) = \sum_i P(\{s_i\}) = 0$ which is a contradiction. If $c > 0$ then $1 = P(S) = \sum_i P(\{s_i\}) = \infty$ which is a contradiction.

3. Chapter 2, Problem 8. We will use the fact that for any $B_1, B_2, \ldots$, $\mathbb{P}(\bigcup_i B_i) \leq \sum_i P(B_i)$. (See Excercise 7.) Thus, $P(\bigcup_i A_i^c) \leq \sum_i P(A_i^c) = 0$ since $P(A_i^c) = 1 - P(A_i) = 0$. Hence, $1 - P(\bigcup_i A_i^c) \geq 1$. Now,

$$\begin{aligned} 1 &\geq P(\bigcap_i A_i) \\ &= 1 - P\left(\bigcap_i A_i\right)^c \\ &= 1 - P\left(\bigcup_i A_i^c\right) \\ &\geq 1. \end{aligned}$$

Therefore, $1 = P(\bigcap_i A_i)$.

4. Chapter 2, Problem 10. The only possible outcomes are $\Omega = \{(1,2),(1,3),(2,3),(3,2)\}$. By assumption $\mathbb{P}(\{(1,2),(1,3)\}) = \mathbb{P}(\{(2,3)\}) = \mathbb{P}(\{(3,2)\}) = 1/3$. If you don't switch, you will win if $\omega = (1,2)$ or $\omega = (1,3)$ so the probability of winning is 1/3. If you switch, you win if $\omega = (2,3)$ or $\omega = (3,2)$. In this case, the probability of winning is 2/3.

5. Chapter 2, Problem 18. Use Bayes' theorem.

$$\mathbb{P}(A_2|B) = \frac{\mathbb{P}(B|A_2)\mathbb{P}(A_2)}{\sum_j \mathbb{P}(B|A_j)\mathbb{P}(A_j)} = \frac{.82 \times .50}{(.65 \times .30) + (.82 \times .50) + (.50 \times .20)} = .58.$$