Homework 3: Solutions

Chapter 1, Problem 1.

$$\mathbb{P}(X = x) = \mathbb{P}(X \le x) - \mathbb{P}(X < x) = F(x^+) - F(x^-).$$

The next one is true by definition.

Chapter 3, Problem 4. (a)

$$F(x) = \begin{cases} 0 & x \le 0 \\ x/4 & 0 < x \le 1 \\ 1/4 & 1 < x \le 3 \\ (1/4) + (3/8)(x-3) & 3 < x \le 5 \\ 1 & x > 5. \end{cases}$$

(b)

(b) $F_Y(y) = P(Y \le y) = P(1/X \le y) = P(X > 1/y)$. The three "zones" for x, $0 \le x \le 1$, $1 \le x \le 3$, $3 \le x \le 5$, correspond to the following zones for y: $1 \le y \le \infty$, $1/3 \le y \le 1$ and $1/5 \le y \le 1/3$. So

$$F_Y(y) = P\left(X > \frac{1}{y}\right) = \int_{1/y}^{5} f_X(x)dx = \begin{cases} 0 & y < 0 \\ \frac{3}{8}\left(5 - \frac{1}{y}\right) & \frac{1}{5} \le y \le \frac{1}{3} \quad (3 \le x \le 5) \\ \frac{3}{4} & \frac{1}{3} \le y \le 1 \quad (1 \le x \le 3) \\ \frac{3}{4} + \frac{1}{4}\left(1 - \frac{1}{y}\right) & y \ge 1 \quad (0 \le x \le 1). \end{cases}$$

Chapter 3, Problem 5. If X and Y are independent then

$$f(x,y) = \mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y) = f_X(x)f_Y(y). \qquad (1)$$

On the other hand, suppose that (1) holds. Then

$$\mathbb{P}(X \in A, Y \in B) \;=\; \sum_{x \in A}\sum_{y \in B} f(x,y) = \sum_{x \in A}\sum_{y \in B} f_x(x) f_Y(y)$$

$$\sum_{x \in A} f_X(x) \sum_{y \in B} f_Y(y) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$$

and hence, X and Y are independent.

Chapter 3, Problem 7. Note that $\min\{X, Y\} > z$ if and only if $X > z$ and $Y > z$. Thus,

$$\begin{aligned} 1 - F_Z(z) &= \mathbb{P}(Z > z) \\ &= \mathbb{P}(\min\{X, Y\} > z) = \mathbb{P}(X > z \text{ and } Y > z) \\ &= \mathbb{P}(X > z)\mathbb{P}(Y > z) = (1 - \mathbb{P}(X \le z))(1 - \mathbb{P}(Y \le z)) \\ &= (1 - z)(1 - z) = (1 - z)^2 \end{aligned}$$

and hence $F_Z(z) = 1 - (1 - z)^2$. Therefore, $f_Z(z) = F_Z'(z) = 2(1 - z)$ for $0 < z < 1$.

Chapter 3, Problem 10. Fix A and B and let $A_g = \{x : g(x) \in A\}$ and $B_h = \{y : h(y) \in B\}$. Then,

$$\begin{aligned} \mathbb{P}(g(X) \in A, h(Y) \in B) &= \mathbb{P}(X \in A_g, Y \in B_h) \\ &= \mathbb{P}(X \in A_g)\mathbb{P}(Y \in B_h) = \mathbb{P}(g(X) \in A)\mathbb{P}(h(Y) \in B). \end{aligned}$$

Thus, $g(X)$ is independent of $h(Y)$.

Chapter 3, Problem 12. Let (a, b) denote the range of X and let (c, d) denote the range of Y. Then

$$f_X(x) = \int_c^d f(x, y)dy = g(x) \int_c^d h(y)dy$$

and

$$f_Y(y) = \int_a^b f(x, y)dy = h(y) \int_a^b g(x)dx.$$

Also,

$$1 = \int_a^b f_X(x)dx = \int_a^b g(x)dx \int_c^d h(y)dy. \tag{2}$$

So

$$f_X(x)f_Y(y) = g(x)h(y) \int_c^d h(y)dy \int_a^b g(x)dx = g(x)h(y) = f(x,y)$$

by (2).

Chapter 3, Problem 13. Y has range $(0, \infty)$. And,

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(e^X \le y) = \mathbb{P}(X \le \log(y)) = \int_{-\infty}^{\log(y)} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

Differentiate to get

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-(\log y)^2/2} \left(\frac{\partial \log y}{\partial y} \right) = \frac{1}{y\sqrt{2\pi}} e^{-(\log y)^2/2}.$$

Chapter 3, Problem 14.

$$\begin{aligned} F_R(r) &= P(R \le r) = P\left(\sqrt{X^2+Y^2} \le r\right) \\ &= P(X^2+Y^2 \le r^2) = \frac{\text{area of disc of radius } r}{\text{area of disc of radius } 1} \\ &= \frac{\pi r^2}{\pi} = r^2. \end{aligned}$$

Thus, $f_R(r) = 2r$ for $0 \le r \le 1$.

Chapter 3, Problem 15. Note that $0 \le Y \le 1$. Now,

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(F_X(X) \le y) = \mathbb{P}(X \le F_X^{-1}(y)) = F_X\left(F_X^{-1}(y)\right) = y.$$

So $F_Y(y) = y$ and hence $Y \sim \text{Uniform}(0, 1)$.

Let $X = F^{-1}(U)$ where $U \sim \mathrm{Uniform}(0,1)$. Then,

$$\mathbb{P}(X \le x) = \mathbb{P}(F^{-1}(U) \le x) = \mathbb{P}(U \le F(x)) = F(x)$$

since $\mathbb{P}(U \le c) = c$ for every $0 < c < 1$. So, $\mathbb{P}(X \le x) = F(x)$ which means that $X \sim F$.

Chapter 3, Problem 20. Let $Z = X - Y$. Then $-1 \le Z \le 1$. For $-1 \le z < 0$, let A_z be the triangle with corners at $(0,-z), (1,1), (z+1,1)$. Then

$$F_Z(z) = \mathbb{P}(Z \le z) = \mathbb{P}((X,Y) \in A_z) = (1+z)^2/2$$

and

$$f_Z(z) = 1 + z.$$

For $0 \le z \le 1$, let A_z be the triangle with corners at $(z,0), (1,0), (1,1-z))$. Then

$$F_Z(z) = \mathbb{P}(Z \le z) = \mathbb{P}((X,Y) \notin A_z) = 1 - \frac{(1-z)^2}{2}$$

and

$$f_Z(z) = 1 - z.$$

Thus,

$$f(z) = \begin{cases} 1+z & -1 < z \le 0 \\ 1-z & 0 < z \le 1. \end{cases}$$

Now let $W = X/Y$ and note that $0 < W < \infty$. For $0 < w < 1$, let A_w be the triangle with corners $(0,0), (0,1), (w,1)$. Then

$$F_W(w) = \mathbb{P}(W \le w) = \mathbb{P}((X,Y) \in A_w) = \frac{w}{2}$$

and

$$f_W(w) = \frac{1}{2}.$$

For $w > 1$, let A_w be the triangle with corners $(0,0), (1,0), (1,1/w)$. Then

$$F_W(w) = \mathbb{P}(W \le w) = \mathbb{P}((X,Y) \notin A_w) = 1 - \frac{1}{2w}$$

and

$$f_W(w) = \frac{1}{2w^2}.$$

So,

$$f(w) = \begin{cases} 1/2 & 0 < w < 1 \\ 1/(2w^2) & w \ge 1. \end{cases}$$

Chapter 3, Problem 18.

(a) .84

(b) .89

(c) 9.58

(d) .37

(e)

$$\begin{aligned} \mathbb{P}(|X| > |x|) &= \mathbb{P}(X > |x|) + \mathbb{P}(X < -|x|) \\ &= \mathbb{P}\left(Z > \frac{|x| - 3}{4}\right) + \mathbb{P}\left(Z < \frac{-|x| - 3}{4}\right) \\ &= 1 - \Phi\left(\frac{|x| - 3}{4}\right) + \Phi(\left(\frac{-|x| - 3}{4}\right). \end{aligned}$$

This is equal to $.05$ when $|x| \approx 9.6$.