Homework 4: Solutions

Chapter 4, Problem 1.

$$\mathbb{E}(Y) = c\prod_{i=1}^{n}\mathbb{E}(X_i) = c\prod_{i=1}^{n}\left(\frac{1}{2}\frac{1}{2} + 2\frac{1}{2}\right) = c\left(\frac{5}{4}\right)^n.$$

Chapter 4, Problem 3.

$$\begin{aligned} F(y) &= \mathbb{P}(Y_n \leq y) = \mathbb{P}(X_1 \leq y, \cdots, X_n \leq y) \\ &= \prod_{i=1}^{n}\mathbb{P}(X_i \leq y) = \prod_{i=1}^{n} y = y^n. \end{aligned}$$

Hence, $f(y) = ny^{n-1}$ and

$$\mathbb{E}(Y_n) = \int_0^1 yf(y)dy = n\int_0^1 yy^{n-1}dy = n\int_0^1 y^n dy = \frac{n}{n+1}.$$

Chapter 4, Problem 4. $X_n = \sum_{i=1}^n W_i$ where $\mathbb{P}(W_i = -1) = p$ and $\mathbb{P}(W_i = 1) = 1-p$. So $\mathbb{E}(W_i) = p(-1)+(1-p)(1) = 1-2p$, $\mathbb{E}(W_i^2) = p(1)+(1-p)(1) = 1$ and $\mathbb{V}(W_i) = 1-(1-2p)^2 = 4p(1-p)$. Hence, $\mathbb{E}(X_n) = n(1-2p)$ and $\mathbb{V}(X_n) = 4np(1-p)$.

Chapter 4, Problem 6.

$$\begin{aligned} \mathbb{E}(Y) &= \sum_y y\mathbb{P}(Y = y) \\ &= \sum_y y\mathbb{P}(r(X) = y) \\ &= \sum_y y \sum_{x \in r^{-1}(y)} \mathbb{P}(X = x) \\ &= \sum_y \sum_{x \in r^{-1}(y)} y\mathbb{P}(X = x) \end{aligned}$$

$$\begin{aligned}
&= \sum_y \sum_{x \in r^{-1}(y)} r(x)\mathbb{P}(X = x) \\
&= \sum_x r(x)\mathbb{P}(X = x) \\
&= \sum_x r(x) f(x).
\end{aligned}$$

Chapter 4, Problem 8. In class we proved that $\mathbb{E}(\overline{X}_n) = \mu$ and $\mathbb{V}(\overline{X}_n) = \sigma^2/n$. For S_n^2, we first write

$$S_n^2 = \frac{\sum_{i=1}^n X_i^2 - n\overline{X}_n^2}{n-1}.$$

Now

$$\mathbb{E}\left(\sum_{i=1}^n X_i^2\right) = \sum_{i=1}^n \mathbb{E}(X_i^2) = \sum_{i=1}^n (\mu^2 + \sigma^2) = n(\mu^2 + \sigma^2).$$

Next,

$$\begin{aligned}
\mathbb{E}(\overline{X}_n^2) &= \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2 \\
&= \frac{1}{n^2}\mathbb{E}\left(\sum_{i=1}^n X_i\right)^2 \\
&= \frac{1}{n^2}\mathbb{E}\left(\sum_{i=1}^n X_i^2 + \sum_{i \neq j} X_i X_j\right) \\
&= \frac{1}{n^2}\left(\sum_{i=1}^n \mathbb{E}(X_i^2) + \sum_{i \neq j} \mathbb{E}(X_i)\mathbb{E}(X_j)\right) \\
&= \frac{1}{n^2}\left(\sum_{i=1}^n (\mu^2 + \sigma^2) + \sum_{i \neq j} \mu^2\right) \\
&= \frac{1}{n^2}\left(n(\mu^2 + \sigma^2) + n(n-1)\mu^2\right).
\end{aligned}$$

Therefore,

$$\mathbb{E}(S_n^2) = \frac{n(\mu^2 + \sigma^2) - \frac{n}{n^2}\left(n(\mu^2 + \sigma^2) + n(n-1)\mu^2\right)}{n-1} = \sigma^2.$$

Chapter 4, Problem 16.

$$\mathbb{E}(r(X)s(Y)|X=x) = \int r(x)s(y)f(y|x)dy = r(x)\int s(y)f(y|x)dy = r(x)\mathbb{E}(s(Y)|X=x).$$

Therefore, $\mathbb{E}(r(X)s(Y)|X) = r(X)\mathbb{E}(s(Y)|X)$. Take $s(Y) = 1$ to get the next result.

Chapter 4, Problem 17. Define m and $b(X)$ as in the hint. Then

$$\begin{aligned}
\mathbb{V}(Y) &= \mathbb{E}((Y - b(X)) + (b(X) - m))^2 \\
&= \mathbb{E}(Y - b(X))^2 + \mathbb{E}(b(X) - m)^2 + 2\mathbb{E}((Y - b(X))(b(X) - m)) \\
&\equiv I + II + III.
\end{aligned}$$

Now

$$I = \mathbb{E}(Y - b(X))^2 = \mathbb{E}\Big(\mathbb{E}(Y - b(X))^2|X\Big) = \mathbb{E}\mathbb{V}(Y|X)$$

since $b(X) = E(Y|X)$. Next,

$$II = \mathbb{E}(b(X) - m)^2 = \mathbb{V}(b(X)) = \mathbb{V}\mathbb{E}(Y|X).$$

Finally

$$\begin{aligned}
III &= 2\mathbb{E}((Y - b(X))(b(X) - m)) \\
&= 2\mathbb{E}\Big(\mathbb{E}((Y - b(X))(b(X) - m))|X\Big) \\
&= 2\mathbb{E}\Big((b(X) - m)\mathbb{E}((Y - b(X))|X\Big) \\
&= 2\mathbb{E}\Big((b(X) - m)(\mathbb{E}(Y|X) - \mathbb{E}(b(X)|X))\Big) \\
&= 2\mathbb{E}\Big((b(X) - m)(b(X) - b(X))\Big) \\
&= 0.
\end{aligned}$$