Homework 5: Solutions

Chapter 5, Problem 1. Since $f_X(x) = (1/\beta)e^{-x/\beta}$,

$$\begin{aligned}\mathbb{E}(X) &= \int x\beta^{-1}e^{-x/\beta}dx = \beta \\ \mathbb{E}(X^2) &= \int x^2\beta^{-1}e^{-x/\beta}dx = 2\beta^2 \\ \mathbb{V}(X) &= 2\beta^2 - \beta^2 = \beta^2.\end{aligned}$$

$$\begin{aligned}\mathbb{P}(|X-\mu_X| \geq k\sigma_X) &= \mathbb{P}(|X-\beta| \geq k\beta) \\ &= \mathbb{P}(X-\beta \geq k\beta) = \mathbb{P}(X \geq (k+1)\beta) \\ &= \int_{\beta(1+k)}^{\infty} f_X(x)\,dx \\ &= e^{-(1+k)}.\end{aligned}$$

Chebyshev's inequality gives,

$$\mathbb{P}(|X-\mu_X| \geq k\sigma_X) \leq \frac{\sigma_X^2}{k^2\sigma_X^2} = \frac{1}{k^2}.$$

Chapter 5, Problem 2. $E(X) = V(X) = \lambda$. So,

$$\begin{aligned}\mathrm{P}\left(X \geq 2\lambda\right) &= \mathrm{P}\left(X - \lambda \geq \lambda\right) \\ &\leq \mathrm{P}\left(|X-\lambda| \geq \lambda\right) \\ &\leq \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}.\end{aligned}$$

Chapter 5, Problem 4. If $p \notin C_n$ then $|p - \widehat{p}_n| > \epsilon_n$. Hence,

$$\begin{aligned}\mathbb{P}(p \notin C_n) &= \mathbb{P}(|p-\widehat{p}_n| > \epsilon_n) \\ &\leq 2e^{-2n\epsilon^2} \quad \text{Hoeffding's inequality} \\ &= \alpha\end{aligned}$$

where the last line follows from plugging in the definition of ϵ_n.

Chapter 6, Problem 1. (a) We showed this in a previous homework. For completeness, here it is. Let us write

$$S_n^2 = \frac{\sum_i (X_i - \overline{X}_n)^2}{n-1} = \frac{\sum_i X_i^2 - n\overline{X}_n^2}{n-1}.$$

Now, $\mathbb{E}\sum_i X_i^2 = nE(X_1)^2$. To compute $\mathbb{E}\left(\overline{X}_n^2\right)$ we will make use of the following fact. If $a_1, \ldots, a_n$ are real numbers then $(\sum_i a_i)^2 = \sum_i a_i^2 + \sum_{i\neq j} a_i a_j$ and the second sum has $n^2 - n = n(n-1)$ terms. So,

$$\begin{aligned}
\mathbb{E}\left(\overline{X}_n^2\right) &= \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2 \\
&= \frac{1}{n^2}\mathbb{E}\left(\sum_{i=1}^n X_i\right)^2 \\
&= \frac{1}{n^2}\mathbb{E}\left(\sum_{i=1}^n X_i^2 + \sum_{i\neq j} X_i X_j\right) \\
&= \frac{1}{n^2}\left(\sum_{i=1}^n \mathbb{E}(X_i)^2 + \sum_{i\neq j} E(X_i X_j)\right) \\
&= \frac{1}{n^2}\left(\sum_{i=1}^n \mathbb{E}(X_i)^2 + \sum_{i\neq j} E(X_i)E(X_j)\right) \\
&= \frac{1}{n^2}\left(\sum_{i=1}^n \mathbb{E}(X_1)^2 + \sum_{i\neq j} \mu^2\right) \\
&= \frac{1}{n^2}\left(nE(X_1)^2 + n(n-1)\mu^2\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{E}(S_n^2) &= \frac{n\mathbb{E}(X_1^2) - n\frac{\left(n\mathbb{E}(X_1)^2 + n(n-1)\mu^2\right)}{n^2}}{n-1} \\
&= \mathbb{E}(X_1^2) - \mu^2 = \sigma^2.
\end{aligned}$$

(b) We can write

$$S_n^2 = \left(\frac{n}{n-1}\right)\frac{1}{n}\sum_{i=1}^n X_i^2 - \left(\frac{n}{n-1}\right)\overline{X}_n^2.$$

By the law of large numbers,

$$\frac{1}{n}\sum_{i=1}^n X_i^2 \xrightarrow{p} \mathbb{E}(X_1^2).$$

Since $(n-1)/n \to 1$, we also have that

$$\left(\frac{n}{n-1}\right)\frac{1}{n}\sum_{i=1}^n X_i^2 \xrightarrow{\text{P}} \mathbb{E}(X_1^2).$$

Also, by the law of large numbers, $\overline{X}_n \xrightarrow{\text{P}} \mu$. Since $g(y) = y^2$ is a continuous function, $\overline{X}_n^2 \xrightarrow{\text{P}} \mu^2$. So,

$$\left(\frac{n}{n-1}\right)\frac{1}{n}\sum_{i=1}^n X_i^2 - \left(\frac{n}{n-1}\right)\overline{X}_n^2 \xrightarrow{\text{P}} \mathbb{E}(X_1^2) - \mu^2 = \sigma^2.$$

Chapter 6, Problem 2. Let $\mu_n = E(X_n)$. Then,

$$\begin{aligned}
\mathbb{E}(X_n - b)^2 &= \mathbb{E}(X_n - \mu_n + \mu_n - b)^2 \\
&= \mathbb{E}\left[(X_n - \mu_n)^2 + (\mu_n - b)^2 + 2(X_n - \mu_n)(\mu_n - b)\right] \\
&= \mathbb{E}(X_n - \mu_n)^2 + (\mu_n - b)^2 + 2(\mu_n - b)\mathbb{E}(X_n - \mu_n) \\
&= \mathbb{E}(X_n - \mu_n)^2 + (\mu_n - b)^2 \\
&= \mathbb{V}(X_n) + (\mu_n - b)^2.
\end{aligned}$$

From this last expression we see that if $\mu_n \to b$ and $\mathbb{V}(X_n) \to 0$ then $\mathbb{E}(X_n - b)^2 \to 0$. Conversely, if $\mathbb{E}(X_n - b)^2 \to 0$ then $\mathbb{V}(X_n) + (\mu_n - b)^2 \to 0$. Hence, $\mathbb{V}(X_n) \to 0$ and $(\mu_n - b)^2 \to 0$.

Chapter 6, Problem 4. Fix $\epsilon > 0$. Then, for all $n > 1/\epsilon$, $\mathbb{P}(|X_n| > \epsilon) = \mathbb{P}(X_n = n) = 1/n^2$. Thus, $\mathbb{P}(|X_n| > \epsilon) \to 0$ and so $X_n \xrightarrow{\text{P}} 0$. But $\mathbb{E}((X_n - 0)^2) = \mathbb{E}(X_n^2) = (1/n^2)\times(1-(1/n^2))+(n^2)\times(1/n^2) = (1/n^2)-(1/n^4)+1 \not\to 0$ so X_n does not converge in quadratic mean.

Chapter 6, Problem 5. $\mathbb{E}(X_i^2) = (1^2 \times p) + (0^2 \times (1 - p)) = p$. $\mathbb{E}(X_i^4) = (1^4 \times p)+(0^4 \times (1-p)) = p$. So $\mathbb{V}(X_i^2) = \mathbb{E}(X_i^4) - \Big(\mathbb{E}(X_i^2)\Big)^2 = p - p^2 = p(1-p)$. So,

$$\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = p$$

and

$$\mathbb{V}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{p(1-p)}{n} \to 0.$$

By problem 2,

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{\text{qm}} p.$$

Since convergence in quadratic mean implies convergence in probability,

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{\text{P}} p.$$

Chapter 6, Problem 6. By the central limit theorem,

$$\overline{X}_n \approx N\left(\mu, \frac{\sigma^2}{n}\right) = N\left(68, \frac{16}{100}\right) = N(68, .16).$$

So,

$$\mathbb{P}(\overline{X}_n > 68) = \mathbb{P}\left(\frac{\overline{X}_n - 68}{.4} > \frac{68 - 68}{.4}\right) = \mathbb{P}\left(\frac{\overline{X}_n - 68}{.4} > 0\right) \approx \mathbb{P}(Z > 0) = \frac{1}{2}.$$

Chapter 6, Problem 9.

$$\mathbb{P}(|X_n - X| > \epsilon) = \frac{1}{n} \to 0$$

and hence $X_n \xrightarrow{\text{P}} X$. Now $X_n - X = 0$ when $X_n = X$. Otherwise $|X_n - X| \geq e^n - 1$. Hence, $E(X_n - X)^2 \geq (e^n - 1)^2(1/n) \to \infty$.

Chapter 6, Problem 13. Let F be the CDF of Z. Then, the CDF of X_n is

$$\begin{aligned}
\mathbb{P}(X_n \leq x) &= \mathbb{P}(n \min\{Z_1, \ldots, Z_n\} \leq x) \\
&= \mathbb{P}\left(\min\{Z_1, \ldots, Z_n\} \leq \frac{x}{n}\right) \\
&= 1 - \mathbb{P}\left(\min\{Z_1, \ldots, Z_n\} > \frac{x}{n}\right) \\
&= 1 - \mathbb{P}\left(Z_1 > \frac{x}{n}, \cdots, Z_n > \frac{x}{n}\right) \\
&= 1 - \prod_{i=1}^{n} \mathbb{P}\left(Z_i > \frac{x}{n}\right) \\
&= 1 - \prod_{i=1}^{n} \left(1 - F\left(\frac{x}{n}\right)\right) \\
&= 1 - \left(1 - F\left(\frac{x}{n}\right)\right)^n \\
&= 1 - \exp\left\{n \log\left(1 - F(x/n)\right)\right\} \\
&= 1 - \exp\left\{\frac{\log\left(1 - F(x/n)\right)}{\frac{1}{n}}\right\} \\
&\to 1 - e^{-x\lambda}
\end{aligned}$$

by L'Hopital's rule. The density of X_n is thus $\lambda e^{-x\lambda}$.