Homework 9: Partial Solutions

Chapter 11, Problem 1. Let F be the cdf of T under H_0. The p-value is $P = 1 - F(T)$. We proved earlier that $F(T) \sim \mathrm{Uniform}(0,1)$. Hence,

$$\mathbb{P}(P < c) = \mathbb{P}(1-F(T) < c) = \mathbb{P}(F(T) > 1-c) = 1-\mathbb{P}(F(T) < 1-c) = 1-(1-c) = c$$

so $P \sim \mathrm{Uniform}(0,1)$.

Chapter 11, Problem 2. Straightforward.

Chapter 11, Problem 3. (a) The rejection region is $R = \{Y > c\}$. The power function is

$$\beta(\theta) = P_\theta(Y > c) = \begin{cases} 1 - \left(\frac{c}{\theta}\right)^n & \text{if c} < \theta \\ 0 & \text{otherwise.} \end{cases}$$

(b) $\beta(1/2) = .05$ if $c = (.95)^{1/n}/2$.

(c) Let P denote a $\mathrm{Uniform}(0,1/2)$ distribution. The p-value is

$$\begin{aligned} P(Y > .48) &= P(\max\{X_1,...,X_n\} > .48) \\ &= 1 - P(\max\{X_1,...,X_n\} < .48) \\ &= 1 - \prod_{i=1}^{20} P(X_i < .48) \\ &= 1 - \prod_{i=1}^{20} .96 \\ &= 1 - .96^{20} = .56. \end{aligned}$$

Insufficient evidence to reject.

(d) The p-value is 0 since all X_i must be less than or equal to 1/2 under H_0. Reject H_0.

Chapter 11, Problem 4. $\widehat{p} = 922/1919 = .48$ with estimated standard error $\widehat{se} = \sqrt{\widehat{p}(1-\widehat{p})/n} = .011$. The Wald test statistic is $(.48 - .50)/.011 = -1.818$ and the p-value is $P(|Z| > 1.818) = 2P(Z > 1.818) = .07$ which is moderate evidence against the null $H_0 : p = 1/2$. A 95 per cent confidence interval is $\widehat{p} \pm 2\,\widehat{se} = (.46, .50)$. The results are equivocal. I'm not convinced.

Chapter 11, Problem 5. (a) The Wald statistic is

$$Z = \frac{\overline{X} - \overline{Y}}{\sqrt{\frac{s_1^2}{8} + \frac{s_2^2}{10}}} = 3.7.$$

The p-value is $P(|Z| > 3.7) = 2P(Z > 3.7) = .0002$ which is strong evidence against the null hypothesis of no difference. The confidence interval is $\overline{X} - \overline{Y} \pm 2\sqrt{\frac{s_1^2}{8} + \frac{s_2^2}{10}} = (.01, .03)$. Going only on the test, one would conclude that these are two different authors since the means are different. But the confidence interval shows that, while the difference is statistically significant, the difference is small. Taken together I would not conclude the authors are different though you might have come to a different conclusion.

(5b) A permutation test on the absolute difference of means yields a p-value of .024. (This depends on your simulation so everyone will have a slightly different number.) The evidence against the null seems a bit weaker.

Chapter 11, Problem 6. (a) Under H_0 $\theta = 0$ and so $T \sim N(0, 1/n)$ and $Z = \sqrt{n}T \sim N(0,1)$. Hence, $P(T > c) = P(\sqrt{n}T > \sqrt{n}c) = P(Z > \sqrt{n}c) = 1 - \Phi(\sqrt{n}c) = \alpha$ implies $c = \Phi^{1-\alpha}/\sqrt{n} = z_\alpha/\sqrt{n}$.

(b) Under $H_1, T \sim N(1, 1/n)$ and $W = \sqrt{n}(T-1) \sim N(0,1)$. Hence,

$$\beta(1) \quad = \quad P_{\theta=1}(T > c)$$

$$
\begin{aligned}
&= P_{\theta=1}\left(\sqrt{n}(T-1) > \sqrt{n}(c-1)\right) \\
&= P\left(W > \sqrt{n}(c-1)\right) \\
&= P\left(W > z_\alpha - \sqrt{n}\right).
\end{aligned}
$$

(c) $z_\alpha - \sqrt{n} \to -\infty$ and hence $\beta(1) \to 1$.

Chapter 11, Problem 6. Under H_1, $W = (\widehat{\theta} - \theta_1)/\widehat{se} \to N(0,1)$. So

$$
\begin{aligned}
\beta(\theta_1) &= P_{\theta_1}(|Z| > z_{\alpha/2}) \\
&= P_{\theta_1}(Z > z_{\alpha/2}) + P_{\theta_1}(Z < -z_{\alpha/2}) \\
&= P_{\theta_1}\left(\frac{\widehat{\theta} - \theta_0}{\widehat{se}} > z_{\alpha/2}\right) + P_{\theta_1}\left(\frac{\widehat{\theta} - \theta_0}{\widehat{se}} < -z_{\alpha/2}\right) \\
&= P_{\theta_1}\left(\widehat{\theta} > \theta_0 + \widehat{se}\, z_{\alpha/2}\right) + P_{\theta_1}\left(\widehat{\theta} < \theta_0 - \widehat{se}\, z_{\alpha/2}\right) \\
&= P_{\theta_1}\left(\frac{\widehat{\theta} - \theta_1}{\widehat{se}} > \frac{\theta_0 - \theta_1}{\widehat{se}} + z_{\alpha/2}\right) + P_{\theta_1}\left(\frac{\widehat{\theta} - \theta_1}{\widehat{se}} < \frac{\theta_0 - \theta_1}{\widehat{se}} - z_{\alpha/2}\right) \\
&= P\left(W > \frac{\theta_0 - \theta_1}{\widehat{se}} + z_{\alpha/2}\right) + P\left(W < \frac{\theta_0 - \theta_1}{\widehat{se}} - z_{\alpha/2}\right) \\
&\geq P\left(W > \frac{\theta_0 - \theta_1}{\widehat{se}} + z_{\alpha/2}\right).
\end{aligned}
$$

As $n \to \infty$, $\widehat{se} \to 0$ and since $\theta_1 > \theta_0$, $(\theta_0 - \theta_1)/\widehat{se} \to -\infty$ and hence $\beta(1) \to 1$.

Chapter 11, Problem 8. As I said in class, we did not really cover how to do this. The grader will accept any answer for this. Some posibilities include: a χ^2 test, a likelihood ratio test (from the appendix). We could also do 4 different two-sample binomial tests using a Bonferroni correction.

Chapter 11, Problem 9. (a) For each comparison we can use test $H_0 : p_1 = p_2$

versus $H_1 : p_1 \neq p_2$ using the Wald test

$$Z = \frac{\widehat{p}_1 - \widehat{p}_2}{\sqrt{\frac{\widehat{p}_1(1-\widehat{p}_1)}{n_1} + \frac{\widehat{p}_2(1-\widehat{p}_2)}{n_2}}}.$$

(We could also use a permutation test.) The results of the Wald test are

Drug	test statistic	odd ratio	p-value
Chlorpromazine	-2.76	2.42	.006
Dimenhydrinate	.64	.82	.520
Pentobarbital (100 mg)	-.48	1.18	.627
Pentobarbital (150 mg)	-1.65	1.67	.100

The only significant result is Chlorpromazine with an odds ratio of 2.42.

(b) There were 4 test so for Bonferroni we should use a significance level of $.05/4 = .0125$. The first result is still significant.

Chapter 11, Problem 10. The mle is $\widehat{\lambda} = \overline{X}$ and the Fisher information is $I(\lambda) = 1/\lambda$ giving the estimated standard error $\widehat{se} = 1/\sqrt{nI(\widehat{\lambda})} = \sqrt{\overline{X}/n}$. The Wald test is to reject H_0 when $|Z| > z_{\alpha/2}$ where

$$Z = \frac{\overline{X} - \lambda_0}{\widehat{se}} = \sqrt{n}\frac{(\overline{X} - \lambda_0)}{\sqrt{\overline{X}}}.$$