Homework 3
36-705
Due via Gradescope on Thursday Sept 17th by 3:00pm

1. χ^2 **tail bound:** In this question, we will prove the χ^2 tail bound from lecture.
   - Let $Z \sim N(0,1)$ and $X = Z^2$. Show that when $t < 1/2$,
   $$M_{X-1}(t) = \frac{\exp(-t)}{\sqrt{1-2t}}.$$
   What happens to the mgf when $t > 1/2$? Is X sub-Gaussian?
   - **Chernoff:** Use the Chernoff method to show that,
   $$\mathbb{P}(X - 1 \geq u) \leq \exp(-u^2/8) \quad \text{for} \;\; u \leq 1.$$
   You may use that fact that $\frac{\exp(-t)}{\sqrt{1-2t}} \leq \exp(2t^2)$ for $|t| \leq 1/4$. Show that
   $$\mathbb{P}(|X - 1| \geq u) \leq 2\exp(-u^2/8) \quad \text{for} \;\; u \leq 1.$$
   - **Sums of** χ^2 **RVs:** Define the RV
   $$X_k = \frac{1}{k}\sum_{i=1}^{k} Z_i^2,$$
   where the Z_i are i.i.d. $N(0,1)$. Show that,
   $$\mathbb{P}(|X_k - 1| \geq u) \leq 2\exp(-ku^2/8) \quad \text{for} \;\; u \leq 1.$$
   Hint: The most straightforward way to do this is to calculate the mgf for $X_k - 1$ and repeat the previous steps.

2. Show that a sequence $X_1, \ldots, X_n$ converges in quadratic mean to a constant c if and only if $\mathbb{E}[X_n] \to c$ and $\mathrm{Var}(X_n) \to 0$, as $n \to \infty$.

3. Let $X_1, \ldots, X_n$ and X be bounded random variables, i.e. for some constant $M > 0$, we have that $|X| \leq M$ and for all i, $|X_i| \leq M$. Show that in this special case, convergence in probability implies convergence in quadratic mean.

4. Show that the following RVs converge in probability to 1.
   - $Y_n = 1 + nX_n$ where $X_n \sim \mathrm{Ber}(1/n)$.
   - $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ where $X_i \sim N(0,1)$.

5. Let $X_n = \mathrm{Ber}(1/2 + 1/n)$, and let $X = \mathrm{Ber}(1/2)$. Does X_n converge to X in distribution? Does it converge in probability?