Homework 4
36-705
Due: Thursday September 24 by 3pm

1. Recall that the Rademacher complexity for a class of functions is
$$R_n(\mathcal{F}) = \mathbb{E}_{\epsilon,X} \sup_{f\in\mathcal{F}} \left| \frac{1}{n}\sum_{i=1}^{n} f(X_i)\epsilon_i \right|.$$
Let
$$\mathcal{F} = \Big\{ f : f(x) = \langle \beta, x\rangle, \|\beta\|_2 \le B \Big\}.$$
Suppose that each $X_i \sim N(0, I_d)$ (multivariate Normal). Now show that:
$$R_n(\mathcal{F}) \le B\sqrt{\frac{d}{n}}.$$
Hint: From Jensen's inequality: $\mathbb{E}[X] \le \sqrt{\mathbb{E}[X^2]}$.

2. Suppose that we take a collection of sets $\mathcal{A}$, and a collection of sets $\mathcal{B}$, and define $\mathcal{C}$ as:
$$\mathcal{C} = \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}.$$
Show that the shattering number:
$$s(\mathcal{C}, n) \le s(\mathcal{A}, n) \times s(\mathcal{B}, n).$$

3. Suppose instead of taking the union of individual sets, we simply collected all sets to define:
$$\mathcal{C} = \{A : A \in \mathcal{A} \text{ or } A \in \mathcal{B}\}.$$
Show that the shattering number:
$$s(\mathcal{C}, n) \le s(\mathcal{A}, n) + s(\mathcal{B}, n).$$

4. Let p_θ be the density on $\mathbb{R}^2$ that is uniform on a disc of radius θ. Let $X_1, \ldots, X_n \sim p_\theta$.
   (a) Write down the likelihood function.
   (b) Find a minimal sufficient statistic.
   (c) Show that X_1 is not a sufficient statistic.

5. Define a partition of $\mathbb{R}^n$ as follows. Two vectors $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are in the same element of the partition if and only if $L(\theta; x_1, \ldots, x_n) \propto L(\theta; y_1, \ldots, y_n)$. Show that this defines a minimal sufficient partition.