Pitman's estimator is often expressed in a different form.

Theorem 6.10. *In a location problem with one-dimensional parameter, Pitman's estimator can be written as* $\mathrm{E}(\Theta|X=x)$*, where we used a "uniform prior" for* Θ*. That is, the MRE estimator for squared-error loss is the formal Bayes rule with respect to Lebesgue measure, if it has finite risk.*

PROOF. The proof consists of showing that the posterior risk of an equivariant rule equals its conditional risk given $Y=(X_1-X_n,\ldots,X_{n-1}-X_n)$. Hence, the estimator that minimizes the posterior risk (namely the posterior mean) will be the same as the estimator that minimizes the conditional risk given Y (namely the MRE as in the proof of Theorem 6.8).

Write $f_{X|\Theta}(x|\theta)=g(x-\theta\mathbf{1})$. Transform X to $(Y^\top,X_n)^\top$. The Jacobian of this transformation is 1, and we get

$$f_{Y,X_n|\Theta}(y,x_n|\theta)=g(y+(x_n-\theta)\mathbf{1},x_n-\theta).$$

The marginal density of Y is

$$f_{Y|\Theta}(y|\theta)=\int g(y+(x_n-\theta)\mathbf{1},x_n-\theta)dx_n=\int g(y+u\mathbf{1},u)du.$$

This does not depend on θ because Y is ancillary (see Problem 4 on page 389), so we will call it $f_Y(y)$. We can write the conditional density

$$f_{X_n|Y,\Theta}(x_n|y,\theta)=\frac{g(y+(x_n-\theta)\mathbf{1},x_n-\theta)}{f_Y(y)}.$$

By letting $\psi=x_n-\theta$, we discover that $\int f_{X_n|Y,\Theta}(x_n|y,\theta)d\theta=1$, so $f_{\Theta|X_n,Y}(\theta|x_n,y)=g(y+(x_n-\theta)\mathbf{1},x_n-\theta)/f_Y(y)$.

Now, write the posterior risk of an arbitrary equivariant estimator $\delta(x)=x_n+w(y)$ as

$$\int(\theta-\delta(x))^2\frac{g(y+(x_n-\theta)\mathbf{1},x_n-\theta)}{f_Y(y)}d\theta=\int(-\psi-w(y))^2\frac{g(y+\psi\mathbf{1},\psi)}{f_Y(y)}d\psi,$$

where we have let $\psi=x_n-\theta$ once again. If we replace ψ by x_n on the right-hand side of this last equation, we recognize it as the conditional risk of δ given $Y=y$ at $\theta=0$. □

The "uniform prior," or Lebesgue measure is closely related to location equivariance. The relation is that the Lebesgue measure of a set is invariant under location shifts. When we deal with more general types of equivariance, a generalization of Theorem 6.10 will emerge in that the MRE estimator will be the formal Bayes rule with respect to a "prior" distribution that is invariant in the appropriate sense.

Example 6.11 (Continuation of Example 6.9; see page 347). Let $X_1,\ldots,X_n$ be IID $N(\theta,1)$ given $\Theta=\theta$. If $X=(X_1,\ldots,X_n)$ and $\delta_0(X)=X_n$, the posterior from a uniform prior is $\Theta\sim N(\overline{x},1/n)$, and the MRE is $\overline{X}$, as we saw earlier.

Theorem 6.19. *Under the same conditions as in Theorem 6.18, the MRE estimator can be written as the formal Bayes rule with respect to a prior having Radon–Nikodym derivative* $1/\theta$ *with respect to Lebesgue measure, if it has finite risk.*

PROOF. As in the proof of Theorem 6.10, we will show that the posterior risk of an equivariant estimator is the same as the conditional risk given the Y from Theorem 6.18. This implies that the formal Bayes rule will also be MRE. Let $f_{X|\Theta}(x|\theta) = g(x/\theta)\theta^{-n}$. The transformation from x to $(y^\top, x_n)^\top$ has Jacobian $|x_n|^{n-1}$, so

$$f_{Y,X_n|\Theta}(y, x_n|1) = g\left(\frac{|x_n|}{\theta}y\right)\frac{|x_n|^{n-1}}{\theta^n},$$

both for $y_n = +1$ and $y_n = -1$. The marginal density of Y is

$$f_Y(y) = \int g\left(\frac{|x_n|}{\theta}y\right)\frac{|x_n|^{n-1}}{\theta^n}dx_n = \int_0^\infty g(uy)u^{n-1}du,$$

and we see that Y is ancillary. So, the conditional density of X_n given $Y = y$ is

$$f_{X_n|Y,\Theta}(x_n|y,\theta) = \frac{g\left(\frac{|x_n|}{\theta}y\right)|x_n|^{n-1}}{\theta^n f_Y(y)}, \text{ if } x_n \text{ has the same sign as } y_n.$$

The normalizing constant for the posterior density is

$$\int_0^\infty g\left(\frac{|x_n|}{\theta}y\right)\frac{|x_n|^{n-1}}{\theta^{n+1}}d\theta = \int_0^\infty g(\psi y)\frac{\psi^{n-1}}{|x_n|}d\psi = \frac{f_Y(y)}{|x_n|},$$

where we used the transformation $\psi = |x_n|/\theta$ with Jacobian $|x_n|/\psi^2$. This implies that the posterior density of Θ is $g(|x_n|y/\theta)|x_n|^n/\theta^{n+1}$.

The posterior risk of an equivariant esitmator $\delta(x) = |x_n|w(y)$ is

$$\begin{aligned}\frac{\int(\theta-\delta(x))^2 g\left(\frac{|x_n|}{\theta}y\right)\frac{|x_n|^n}{\theta^{n+3}}d\theta}{f_Y(y)} &= \frac{\int\left(\frac{|x_n|}{\psi}-|x_n|w(y)\right)^2 g(\psi y)\frac{\psi^{n+1}}{|x_n|^2}d\psi}{f_Y(y)} \\ &= \frac{\int(1-\psi w(y))^2 g(\psi y)\psi^{n-1}d\psi}{f_Y(y)},\end{aligned}$$

where we have used the transformation $\psi = |x_n|/\theta$ again. Switching ψ to $|x_n|$ in the last equation, we recognize this as the conditional risk given $Y = y$ and $\Theta = 1$. □

The reader should note that the measure $\lambda(A) = \int_A d\theta/\theta$ is invariant under scale changes. That is, the measure of a set A of positive numbers is the same as the measure of $|c|A$ for all nonzero c.