This product converges to $\mathrm{E}\exp\left\{it^\top X\right\}\mathrm{E}\exp\left\{is^\top Y\right\}$, which is the characteristic function of independent X and Y. Now apply Theorem B.93. $\square$

Using the fact that a constant is independent of everything, the following is a corollary to Theorem 7.20 for Euclidean spaces. The result holds for more general spaces also.

Corollary 7.21.[7] *Suppose that $\{X_n\}_{n=1}^\infty$ take values in a metric space $\mathcal{X}$. If $X_n \xrightarrow{\mathcal{D}} X$ and $b \in \mathcal{Y}$ is a constant, then $(X_n, b) \xrightarrow{\mathcal{D}} (X, b)$.*

The conclusions of the following theorem are taken for granted in many calculations of asymptotic distributions.

Theorem 7.22. *1. Suppose that $\{X_n\}_{n=1}^\infty$ take values in a topological space $\mathcal{X}$ and that $\{Y_n\}_{n=1}^\infty$ take values in a topological space $\mathcal{Y}$. If $(X_n, Y_n) \xrightarrow{\mathcal{D}} (X, Y)$, then $X_n \xrightarrow{\mathcal{D}} X$.*
2. Suppose that $\{X_n\}_{n=1}^\infty$ take values in a metric space $\mathcal{X}$ and that $\{Y_n\}_{n=1}^\infty$ take values in a metric space $\mathcal{Y}$. Let $b \in \mathcal{Y}$. If $X_n \xrightarrow{\mathcal{D}} X$ and $Y_n \xrightarrow{P} b$, then $(X_n, Y_n) \xrightarrow{\mathcal{D}} (X, b)$.

PROOF. For part 1, let $g : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ be defined by $g(x, y) = x$. Then g is continuous and the continuous mapping theorem B.88 says that $X_n = g(X_n, Y_n) \xrightarrow{\mathcal{D}} g(X, Y) = X$.

For part 2, let d_1 be the metric in $\mathcal{X}$ and let d_2 be the metric in $\mathcal{Y}$. Then

$$d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2)$$

is a metric in $\mathcal{X} \times \mathcal{Y}$ and the product σ-field is the Borel σ-field. By Corollary 7.21, we have that $(X_n, b) \xrightarrow{\mathcal{D}} (X, b)$. We have assumed that $d((X_n, Y_n), (X_n, b)) = d_2(Y_n, b) = o_P(1)$. So, by Lemma 7.19, $(X_n, Y_n) \xrightarrow{\mathcal{D}} (X, b)$. $\square$

7.1.3 The Delta Method

A method for finding the asymptotic distribution of a function of a random vector is based on Lemma 7.19 and is called *the delta method*. [See Rao (1973), Chapter 6.] As an example, let Y_n be the average of n IID random variables with mean μ and variance σ^2. The central limit theorem B.97 says that $\sqrt{n}(Y_n - \mu) \xrightarrow{\mathcal{D}} N(0, \sigma^2)$.[8] Now, let g be a function with continuous derivative. We can write

$$g(t) = g(\mu) + (t - \mu)g'(\mu) + o(t - \mu).$$

[7] This corollary is used in the proof of Theorem 7.22.

[8] It is common to call an estimator Z_n, with the property that $\sqrt{n}(Z_n - \theta)$ converges in distribution to a nondegenerate distribution, $\sqrt{n}$*-consistent*.