Assume that J_1 is nonsingular. Also, assume that there exists $H_r(x)$ such that

$$\sup_{\|\theta-\theta_0\|\le r}\left|\frac{\partial}{\partial\theta_t}\psi_j(x,\theta)-\frac{\partial}{\partial\theta_t}\psi_j(x,\theta_0)\right|\le H_r(x),$$

where $\lim_{r\to 0}\mathrm{E}_{\theta_0}H_r(X_i)=0$. *Let* $M(\theta)=((m_{j,t}(\theta)))$, *where*

$$m_{j,t}(\theta)=\frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial\theta_t}\psi_j(X_i,\theta).$$

Let $\hat{\Theta}_n^*=\tilde{\Theta}_n-M^{-1}(\tilde{\Theta}_n)\sum_{i=1}^n\psi(X_i,\tilde{\Theta}_n)/n$. *Then, under* P_{θ_0},

$$\sqrt{n}(\hat{\Theta}_n^*-\theta_0)\xrightarrow{\mathcal{D}}N_k(0,J_1^{-1}J_2J_1^{-1\top}).$$

PROOF. Let $p(\theta)=\sum_{i=1}^n\psi(X_i,\theta)/n$, so that $\hat{\Theta}_n^*=\tilde{\Theta}_n-M^{-1}(\tilde{\Theta}_n)p(\tilde{\Theta}_n)$. We can write

$$p(\tilde{\Theta}_n)=p(\theta_0)+M(\theta^*)(\tilde{\Theta}_n-\theta_0),$$

where θ^* is between θ_0 and $\tilde{\Theta}_n$. Let θ' stand for either θ^* or $\tilde{\Theta}_n$. Let $\epsilon>0$, let $\Delta_{j,t}=m_{j,t}(\theta')-m_{j,t}(\theta_0)$, and let r be small enough so that $\mathrm{E}_{\theta_0}H_r(X_i)<\epsilon/2$. Then $|\Delta_{j,t}|\le\sum_{i=1}^n|H_{\|\theta'-\theta_0\|}(X_i)|/n$ and

$$\begin{aligned}
P'_{\theta_0}(|\Delta_{j,t}|>\epsilon) &\le P'_{\theta_0}\left(\frac{1}{n}\sum_{i=1}^n H_r(X_i)>\epsilon\right)+P'_{\theta_0}(\|\theta'-\theta_0\|\ge r)\\
&\le P'_{\theta_0}\left(\left|\frac{1}{n}\sum_{i=1}^n H_r(X_i)-\mathrm{E}_{\theta_0}H_r(X_1)\right|>\frac{\epsilon}{2}\right)+\\
&\quad P'_{\theta_0}(\|\theta'-\theta_0\|\ge r).
\end{aligned}$$

The weak law of large numbers and the fact that $\theta'\xrightarrow{P}\theta_0$ imply that these last two probabilities go to 0, so $\Delta_{j,t}=o_P(1)$. This means that

$$\begin{aligned}
M(\theta^*) &= M(\theta_0)+o_P(1)=M(\theta_0)[I+o_P(1)],\\
M(\tilde{\Theta}_n) &= M(\theta_0)+o_P(1)=M(\theta_0)[I+o_P(1)].
\end{aligned}$$

Hence $p(\tilde{\Theta}_n)=p(\theta_0)+M(\theta_0)(\tilde{\Theta}_n-\theta_0)+o_P(1/\sqrt{n})$. Since $\mathrm{E}_{\theta_0}\psi(X_i,\theta_0)=0$, we have $p(\theta_0)=\mathcal{O}_P(1/\sqrt{n})$ and it follows that

$$\begin{aligned}
\hat{\Theta}_n^* &= \tilde{\Theta}_n-M(\theta_0)^{-1}\Big(p(\theta_0)+M(\theta_0)(\tilde{\Theta}_n-\theta_0)\Big)+o_P\left(\frac{1}{\sqrt{n}}\right)\\
&= \tilde{\Theta}_n-M(\theta_0)^{-1}p(\theta_0)-(\tilde{\Theta}_n-\theta_0)+o_P\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

Collecting terms, we get

$$\sqrt{n}(\hat{\Theta}_n^*-\theta_0)=-\sqrt{n}M(\theta_0)^{-1}p(\theta_0)+o_P(1). \tag{7.76}$$