we can write

$$\mathrm{E}(g(\Theta)|X=x) = \frac{\int g(\theta) f_\Theta(\theta) f_{X|\Theta}(x|\theta) d\theta}{\int f_\Theta(\theta) f_{X|\Theta}(x|\theta) d\theta}.$$

The method of Laplace provides approximations to each of these integrals for specific values of x. Some conditions and notation are needed to state the approximations precisely.

Theorem 7.108. *For each n, let $(\mathcal{X}_n, \mathcal{B}_n)$ be a Borel space, and let X_n be a random quantity taking values in $\mathcal{X}_n$. Let $X^n = (X_1, \ldots, X_n)$ and let $(\mathcal{X}^n, \mathcal{B}^n)$ be the product space of $\mathcal{X}_1, \ldots, \mathcal{X}_n$. Let $\{P_\theta : \theta \in \Omega\}$ be a parametric family of distributions for $\{X_n\}_{n=1}^\infty$ with $\Omega \subseteq \mathbb{R}$. Suppose that the distribution of X^n given $\Theta = \theta$ is absolutely continuous with respect to a measure ν_n on $(\mathcal{X}^n, \mathcal{B}^n)$ for all n with density $f_{X^n|\Theta}(\cdot|\theta)$. Let $g : \Omega \to \mathbb{R}^+$ be a function. Let $f_\Theta(\theta)$ be the prior density of Θ with respect to Lebesgue measure. Assume that $f_{X^n|\Theta}(x^n|\theta)$ for all n and $x^n \in \mathcal{X}^n$, $g(\theta)$, and $f_\Theta(\theta)$ are all continuously differentiable with respect to θ six times. Assume that $\int g(\theta) f_\Theta(\theta) d\theta < \infty$. Define*

$$\begin{aligned}
\ell_n(\theta; x^n) &= \log f_{X^n|\Theta}(x^n|\theta), \\
H_n(\theta; x^n) &= \frac{1}{n}\left[\ell_n(\theta; x^n) + \log f_\Theta(\theta)\right], \\
H_n^*(\theta; x^n) &= H_n(\theta; x^n) + \frac{1}{n}\log g(\theta).
\end{aligned}$$

Now let $\mathcal{Y} = \prod_{n=1}^\infty \mathcal{X}^n$ and define the set $A \subseteq \mathcal{Y}$ as the set of all $x = (x^1, x^2, x^3, \ldots) \in \mathcal{Y}$ with the following properties:

- *The integrals $\int g(\theta) f_\Theta(\theta) f_{X^n|\Theta}(x^n|\theta) d\theta$ and $\int f_\Theta(\theta) f_{X^n|\Theta}(x^n|\theta) d\theta$ are finite for all n.*
- *ℓ_n achieves its maximum at a point $\theta_n'(x^n)$ for each n.*
- *For each n, H_n and H_n^* achieve their maxima at points $\hat\theta_n(x^n)$ and $\theta_n^*(x^n)$, respectively, where the first derivatives are zero.*
- *$\hat\theta_n(x^n)$ and $\theta_n^*(x^n)$ converge as $n \to \infty$.*
- *The second derivatives of H_n and H_n^* at their maxima converge to negative numbers.*
- *There exist $\delta_0, N_0, M > 0$ such that for all $n \geq N_0$, the absolute values of H_n, H_n^* and their first six derivatives are all bounded by M for $|\theta - \hat\theta_n(x^n)| \leq 2\delta_0$.*
- *For every $\delta > 0$,*

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{|\theta - \hat\theta_n(x^n)| > \delta} \ell_n(\theta; x^n) - \ell_n(\theta_n'(x^n); x^n) < 0. \tag{7.109}$$

For each $(x_1, x_2, x_3, \ldots) \in A$, *define*

$$\sigma_n^2(x^n) = -\frac{1}{H_n''(\hat{\theta}_n(x^n); x^n)}, \quad \sigma_n^{*2}(x^n) = -\frac{1}{H_n^{*''}(\theta_n^*(x^n); x^n)}.$$

For each $(x_1, x_2, x_3, \ldots) \in A$, $\mathrm{E}(g(\Theta)|X^n = x^n)$ *equals*

$$\frac{\sigma_n^*(x^n)}{\sigma_n(x^n)} \exp(n[H_n^*(\theta^*(x^n); x^n) - H_n(\hat{\theta}_n(x^n); x^n)]) \left[1 + \mathcal{O}(n^{-2})\right].$$

PROOF. Since virtually everything depends on n and x^n in the statement of the theorem, we will simplify the notation by not explicitly expressing that dependence. For example, $H(\hat{\theta})$ will stand for $H_n(\hat{\theta}_n(x^n); x^n)$. Now, let $(x_1, x_2, \ldots) \in A$ and write

$$\mathrm{E}(g(\Theta)|X^n = x^n) = \frac{\int_\Omega \exp(nH^*(\theta))d\theta}{\int_\Omega \exp(nH(\theta))d\theta}. \tag{7.110}$$

We assumed that $\hat{\theta}$ and θ^* both converge. We now show that they converge to the same thing and that so does θ'. Suppose that $\hat{\theta}$ converges to θ_0 and θ^* converges to θ_1. If there exists $\delta > 0$ such that $|\theta' - \hat{\theta}| > \delta$ for infinitely many n, then (7.109) says that there is $\eta > 0$ such that $\ell(\hat{\theta}) < \ell(\theta') - \eta$ infinitely often. Since $H(\hat{\theta}) = \ell(\hat{\theta}) + \mathcal{O}(n^{-1})$, and similarly for θ', it follows that $H(\hat{\theta}) < H(\theta') - \eta$, infinitely often. This contradicts the definition of $\hat{\theta}$ being the location of the maximum of H for all n. It follows that, for each $\delta > 0$, there exists N such that $n \geq N$ implies $|\theta' - \hat{\theta}| < \delta$. Hence θ' converges to θ_0 also. A similar argument shows that θ' converges to θ_1, hence $\theta_0 = \theta_1$. Let $\Omega' = \Omega \cap (\theta_0 - \delta_0, \theta_0 + \delta_0)$. Since $\exp(nH(\theta)) = \exp(\ell(\theta))f_\Theta(\theta)$, condition (7.109) implies that $\int_{\Omega \setminus \Omega'} \exp(nH(\theta))d\theta$ and $\int_{\Omega \setminus \Omega'} \exp(nH^*(\theta))d\theta$ are exponentially small. For this reason, we can replace Ω by Ω' in (7.110) without incurring an error larger than $\mathcal{O}(n^{-2})$.

If we expand $H(\theta)$ in a Taylor series (see Theorem C.1) around $\theta = \hat{\theta}$, we get

$$\begin{aligned} H(\theta) &= H(\hat{\theta}) + (\theta - \hat{\theta})H'(\hat{\theta}) + \frac{1}{2}(\theta - \hat{\theta})^2 H''(\hat{\theta}) + \frac{1}{6}(\theta - \hat{\theta})^3 H'''(\hat{\theta}) \\ &+ \frac{1}{24}(\theta - \hat{\theta})^4 H^{(iv)}(\hat{\theta}) + \frac{1}{120}(\theta - \hat{\theta})^5 H^{(v)}(\hat{\theta}) + \frac{1}{720}(\theta - \hat{\theta})^6 H^{(vi)}(\tilde{\theta}), \end{aligned}$$

where $\tilde{\theta}$ is between θ and $\hat{\theta}$, and $H^{(iv)}$, $H^{(v)}$, and $H^{(vi)}$ respectively stand for the fourth, fifth, and sixth derivatives of H. Use the Taylor series of $\exp(x)$ around $x = 0$ and the fact that $H'(\hat{\theta}) = 0$ to write

$$\begin{aligned} \exp(nH(\theta)) \quad = \quad & \exp(nH(\hat{\theta})) \exp\left(\frac{n}{2}(\theta - \hat{\theta})^2 H''(\hat{\theta})\right) \\ & \times \Big[1 + \frac{n}{6}(\theta - \hat{\theta})^3 H'''(\hat{\theta}) + \frac{n}{24}(\theta - \hat{\theta})^4 H^{(iv)}(\hat{\theta}) \\ & + \frac{n}{120}(\theta - \hat{\theta})^5 H^{(v)}(\hat{\theta}) + \frac{n^2}{72}(\theta - \hat{\theta})^6 H'''(\hat{\theta})^2 + R_n(\theta)\Big], \end{aligned}$$