exists a sequence $\{\epsilon_k\}_{k=1}^{\infty}$ converging to 0 such that $P(d(X,B)=\epsilon_k)=0$ for all k. It follows that $\lim_{n\to\infty} P_n(C_{\epsilon_k}) = P(C_{\epsilon_k})$ for all k. Since $P_n(B) \le P_n(C_{\epsilon_k})$ for every n and k, we have, for every k,

$$\limsup_{n\to\infty} P_n(B) \le \lim_{n\to\infty} P_n(C_{\epsilon_k}) = P(C_{\epsilon_k}).$$

Since $P(B) = \lim_{k\to\infty} P(C_{\epsilon_k})$, we have (2). So, (2), (3), and (4) are equivalent and (1) implies (2).

All that remains is to prove that (2) implies (1). Assume (2), and let f be a bounded continuous function. Let $m < f(x) < M$ for all x. For each k, let $F_{i,k} = \{x : f(x) \le m + (M-m)i/k\}$ for $i = 1,\ldots,k$. Let $F_{0,k} = \emptyset$. Each $F_{i,k}$ is closed, since f is continuous. Let $G_{i,k} = F_{i,k} \setminus F_{i-1,k}$ for $i = 1,\ldots,k$. It is easy to see that for every probability Q,

$$m + (M-m)\sum_{i=1}^{k} \frac{i-1}{k} Q(G_{i,k}) < \int f(x)dQ(x) \le m + (M-m)\sum_{i=1}^{k} \frac{i}{k} Q(G_{i,k}).$$

Since $Q(G_{i,k}) = Q(F_{i,k}) - Q(F_{i-1,k})$ for every i and k, we get

$$M - \frac{M-m}{k}\sum_{i=1}^{k} Q(F_{i,k}) < \int f(x)dQ(x) \le M + \frac{M-m}{k} - \frac{M-m}{k}\sum_{i=1}^{k} Q(F_{i,k}). \tag{B.84}$$

For each i,

$$\limsup_{n\to\infty} P_n(F_{i,k}) \le P(F_{i,k}). \tag{B.85}$$

It follows that, for every k,

$$\begin{aligned}
\int f(x)dP(x) &\le M + \frac{M-m}{k} - \frac{M-m}{k}\sum_{i=1}^{k} P(F_{i,k}) \\
&\le M + \frac{M-m}{k} - \frac{M-m}{k}\sum_{i=1}^{k} \limsup_{n\to\infty} P_n(F_{i,k}) \\
&\le \frac{M-m}{k} + \liminf_{n\to\infty} \int f(x)dP_n(x),
\end{aligned}$$

where the first inequality follows from the second inequality in (B.84) with $Q = P$, the second inequality follows from (B.85), and the third inequality follows from the first inequality in (B.84) with $Q = P_n$. Letting k be arbitrarily large, we get

$$\int f(x)dP(x) \le \liminf_{n\to\infty} \int f(x)dP_n(x). \tag{B.86}$$

Now, apply the same reasoning to $-f$ to get

$$\begin{aligned}
-\int f(x)dP(x) &\le \liminf_{n\to\infty} \int -f(x)dP_n(x) = -\limsup_{n\to\infty} \int f(x)dP_n(x), \\
\int f(x)dP(x) &\ge \limsup_{n\to\infty} \int f(x)dP_n(x). \qquad \text{(B.87)}
\end{aligned}$$

Together, (B.86) and (B.87) imply (1). □